

## CHAPTER 12

# METHOD OF SLOPE-DEFLECTION EQUATIONS

A method of slope-deflection equations is a classical displacement method commonly used in the analysis of statically indeterminate, flexure-dominating structures such as beams and plane frames. In this method, rotations at all nodes and relative transverse displacements between two ends of all members (resulting from the discretization of continuous to discrete structure) termed *sway displacements* are chosen as primary unknowns. For each member, the bending moments at both ends can be expressed in terms of the end rotations and the sway displacement of that member. These two key equations are known as the *slope-deflection equations*. A complete set of equations governing all primary unknowns of the structure can readily be obtained by combining these slope-deflection equations via the enforcement of moment equilibrium at all nodes and force equilibrium associated with the sway displacements. This set of linear algebraic equations is then solved to obtain the rotation at all nodes and the sway displacements. The end moments can subsequently be computed from the slope-deflection equations of each member and all other static quantities (e.g. shear forces, axial forces, and support reactions) can readily be obtained from static equilibrium.

The method of slope-deflection equations established further below is based upon following key assumptions: (i) the structure consists of a collection of straight and prismatic skeletons or members; (ii) all members are made of a homogeneous, isotropic, linearly elastic material; (iii) the deformation (i.e. curvature) is related to the displacement via the linearized (or infinitesimal) kinematics; (iv) the shear deformation is neglected (i.e. the plane section always remains plane and normal to the neutral axis before and after undergoing deformation); (v) the member is inextensible (i.e. the axial deformation of the neutral axis is ignored); and (vi) equilibrium equations are formed based on the geometry of an undeformed configuration of the structure. Following sections present the derivation of the slope-deflection equations, forms of these equations for certain special cases, computation of fixed-end moments, applications of the slope-deflection equations in the analysis of various beam and frame structures, and the treatment of symmetry and anti-symmetry of structures.

## 12.1 Derivation of Slope-deflection Equations

In this section, we present the derivation of the slope-deflection equations of a single straight member. Three basic components for structural mechanics (i.e. static equilibriums, kinematics, and constitutive relations) are utilized along with assumptions described above to obtain such equations.

Consider a straight, prismatic member AB of length  $L$  and moment of inertia  $I$  as shown schematically in Figure 12.1. This member is made of a homogeneous, isotropic, linearly elastic material of Young's modulus  $E$ . For convenience and brevity of references in the development carried out further below, let us define a local coordinate system  $\{x, y, z\}$  for this particular member such that its origin locates at point A, the  $x$ -axis directs along the member, and the  $y$ -axis is oriented such that the  $z$ -axis direct outward from the paper.

The member AB is subjected to arbitrary member loads as shown in Figure 12.1. The axial forces, the shear forces, and the bending moments at end A and end B are denoted by  $\{N_{AB}, V_{AB}, M_{AB}\}$  and  $\{N_{BA}, V_{BA}, M_{BA}\}$ , respectively. It should be noted that  $\{N_{AB}, V_{AB}, M_{AB}, N_{BA}, V_{BA}, M_{BA}\}$  are not all independent but they are related to the member loads via three independent static equilibrium equations; for instance, if  $\{M_A, M_B\}$  are known,  $\{V_A, V_B\}$  can readily be computed from force equilibrium in the  $y$ -direction and moment equilibrium in the  $z$ -direction and if one of  $\{N_A, N_B\}$  is known, the other can be computed from force equilibrium in the  $x$ -direction. Note that in the development presented further below the positive sign convention of the end forces and end moments follows the local coordinate system  $\{x, y, z\}$ . Resulting from applied loads, the member

AB displaces to a new configuration called the *deformed configuration* as shown in Figure 12.1. The displacement in a longitudinal direction (x-direction), the displacement in a transverse direction (y-direction), and the rotation (in z-direction) at any point x are denoted by  $u(x)$ ,  $v(x)$ , and  $\theta(x)$ , respectively. In addition, we define  $u(0) = u_A$ ,  $v(0) = v_A$ ,  $\theta(0) = \theta_A$  and  $u(L) = u_B$ ,  $v(L) = v_B$ ,  $\theta(L) = \theta_B$  as the displacements and rotations at the end A and end B, respectively.

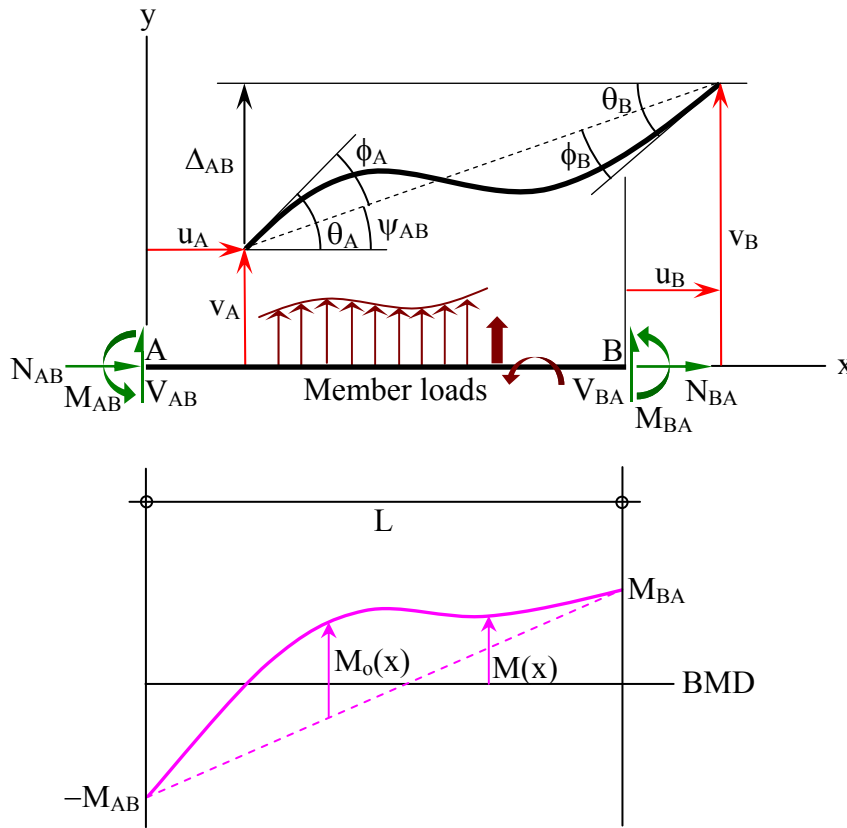


Figure 12.1 Schematic of undeformed and deformed configurations of straight member AB and the corresponding bending moment diagram

The bending moment at any point x, denoted by  $M(x)$ , can be obtained from static equilibrium (along with the assumption (vi)) and the result is given by

$$M(x) = -M_{AB} + (M_{AB} + M_{BA}) \frac{x}{L} + M_o(x) \quad (12.1)$$

where  $M_o(x)$  is the bending moment due to the member loads (i.e. loads acting on the member) in the absence of the end moments (i.e.  $M_o(0) = M_o(L) = 0$ ) or, equivalently, it can be viewed as the bending moment of a simply-supported beam of length L subjected to the same set of member loads. Based on the linearized kinematics, the curvature  $\kappa(x)$  is related to the rotation  $\theta(x)$  and the transverse displacement  $v(x)$  by

$$\kappa(x) = \frac{d\theta}{dx} = \frac{d^2v}{dx^2} \quad (12.2)$$

Upon exploiting the assumptions (ii), (iv), and (v), the bending moment  $M(x)$  can be linearly related to the curvature  $\kappa(x)$  in a form called the “moment-curvature relationship” given below:

$$\kappa(x) = \frac{M(x)}{EI} \quad (12.3)$$

Combining equations (12.1), (12.2), and (12.3) leads to a governing differential equation in terms of the rotation:

$$EI \frac{d\theta}{dx} = -M_{AB} + (M_{AB} + M_{BA}) \frac{x}{L} + M_o(x) \quad (12.4)$$

By integrating equation (12.4) from  $x = 0$  to  $x = \xi$  and then employing the definition  $\theta(0) = \theta_A$ , we obtain

$$EI (\theta(\xi) - \theta_A) = -M_{AB}\xi + (M_{AB} + M_{BA}) \frac{\xi^2}{2L} + \int_0^\xi M_o(x) dx \quad (12.5)$$

Substituting  $\xi = L$  into equation (12.5) along with the definition  $\theta(L) = \theta_B$  leads to a relation among the end rotations, the end moments and the moment due to member loads:

$$EI(\theta_B - \theta_A) = \left( \frac{M_{BA}}{2} - \frac{M_{AB}}{2} \right) L + \int_0^L M_o(x) dx \quad (12.6)$$

By recalling the relation  $\theta(\xi) = dv/d\xi$  and then integrating equation (12.5) from  $\xi = 0$  to  $\xi = L$ , we then obtain

$$EI(v_B - v_A - \theta_A L) = \left( \frac{M_{BA}}{6} - \frac{M_{AB}}{3} \right) L^2 + \int_0^L \int_0^\xi M_o(x) dx d\xi \quad (12.7)$$

where the definitions  $v(0) = v_A$  and  $v(L) = v_B$  have been used. By changing the order of integration of a double integral appearing on the right hand side of equation (12.7), it can readily be reduced to a single integral as indicated below:

$$\int_0^L \int_0^\xi M_o(x) dx d\xi = \int_0^L \int_x^L M_o(x) d\xi dx = \int_0^L M_o(x) \int_x^L d\xi dx = \int_0^L M_o(x)(L-x) dx \quad (12.8)$$

By substituting (12.8) into (12.7), it leads to

$$EI(v_B - v_A - \theta_A L) = \left( \frac{M_{BA}}{6} - \frac{M_{AB}}{3} \right) L^2 + \int_0^L M_o(x)(L-x) dx \quad (12.9)$$

It should be noted that equations (12.6) and (12.9) are, in fact, the curvature area equations relating the end transverse displacements  $\{v_A, v_B\}$  and the end rotations  $\{\theta_A, \theta_B\}$  of the member AB (see Chapter 4). Upon solving these two equations, the end moments  $\{M_{AB}, M_{BA}\}$  can be obtained in a form

$$M_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B) - \frac{6EI}{L}\psi_{AB} + \frac{6}{L^2} \int_0^L M_o(x) \left( \frac{2L}{3} - x \right) dx \quad (12.10)$$

$$M_{BA} = \frac{2EI}{L}(\theta_A + 2\theta_B) - \frac{6EI}{L}\psi_{AB} + \frac{6}{L^2} \int_0^L M_o(x) \left( \frac{L}{3} - x \right) dx \quad (12.11)$$

where  $\psi_{AB}$  is the *sway angle* or chord rotation which can be defined in terms of the sway displacement  $\Delta_{AB} = v_B - v_A$  by

$$\psi_{AB} = \frac{\Delta_{AB}}{L} = \frac{v_B - v_A}{L} \quad (12.12)$$

To interpret the physical meaning of two integrals appearing on the right hand side of equations (12.10) and (12.11), let us consider the special case that both ends of the member AB are fully fixed, i.e.  $\theta_A = \theta_B = 0$  and  $v_A = v_B = 0$ . The end moments for this particular case are termed as the *fixed-end moments* due to member loads and are commonly denoted by  $M_{AB} \equiv FEM_{AB}$  and  $M_{BA} \equiv FEM_{BA}$ . By specializing equations (12.10) and (12.11) to this particular case (by replacing  $\theta_A = \theta_B = 0$  and  $v_A = v_B = 0$ ), we then obtain the expression of the fixed-end moments  $FEM_{AB}$  and  $FEM_{BA}$  as

$$FEM_{AB} = \frac{6}{L^2} \int_0^L M_o(x) \left( \frac{2L}{3} - x \right) dx \quad (12.13)$$

$$FEM_{BA} = \frac{6}{L^2} \int_0^L M_o(x) \left( \frac{L}{3} - x \right) dx \quad (12.14)$$

With the relations (12.13) and (12.14), equations (12.10) and (12.11) now become

$$M_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B) - \frac{6EI}{L}\psi_{AB} + FEM_{AB} \quad (12.15)$$

$$M_{BA} = \frac{2EI}{L}(\theta_A + 2\theta_B) - \frac{6EI}{L}\psi_{AB} + FEM_{BA} \quad (12.16)$$

These two equations are known as *slope-deflection equations* of the member AB. The end moments  $\{M_{AB}, M_{BA}\}$  are expressed in terms of member loads (through the fixed-end moments  $\{FEM_{AB}, FEM_{BA}\}$ ), end rotations  $\{\theta_A, \theta_B\}$  and the sway angle  $\psi_{AB}$ . It is obvious that the slope-deflection equations (12.15) and (12.16) for a particular member contain four groups of information (as indicated in Figure 12.2): the end moments  $\{M_{AB}, M_{BA}\}$ , the end rotations and sway angle  $\{\theta_A, \theta_B, \psi_{AB}\}$ , the contribution of member loads  $\{FEM_{AB}, FEM_{BA}\}$ , and material and member properties  $\{E, I, L\}$ . When applied to a single member, the slope-deflection equations are sufficient to determine two quantities from these groups of information provided that all others are known.

It is worth noting that the longitudinal displacement  $u(x)$  does not involve in the development of the slope-deflection equations; this results mainly from the assumptions (iii) and (vi). However, from the assumption (v) along with the assumption (iii), we readily obtain a simple relation  $u(x) = u_A = u_B$  for all  $x \in [0, L]$ ; this implies that the longitudinal displacement at any point of the member is identical and completely known if the value at one particular point is prescribed.

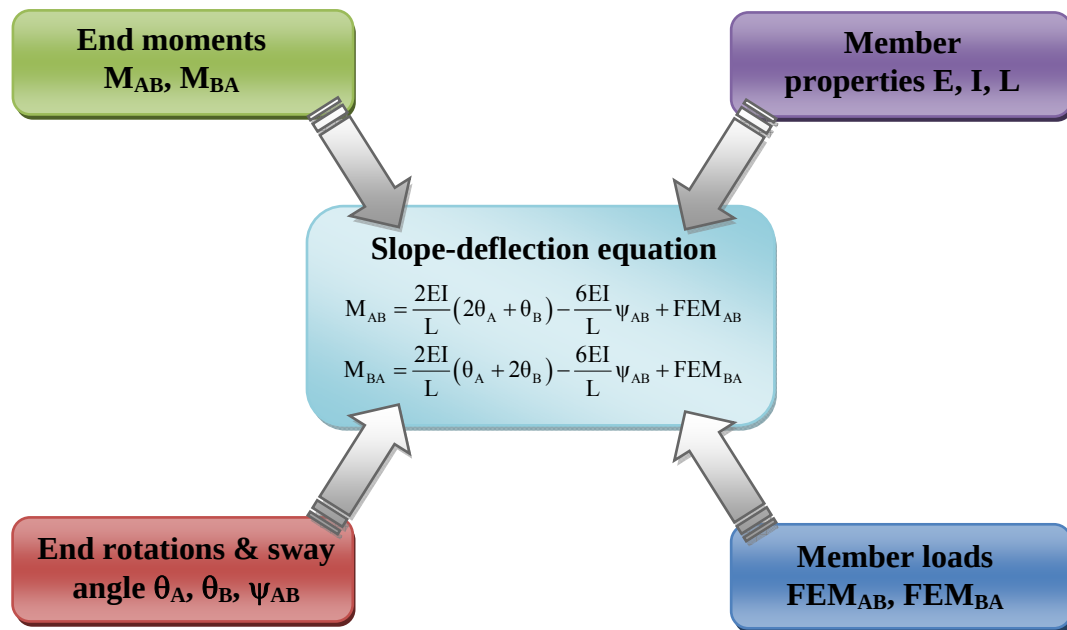


Figure 12.2 Schematic indicating all information involved in slope-deflection equations

## 12.2 Sign Convention

According to the derivation of the slope-deflection equations presented in above section, the end rotations  $\theta_A$  and  $\theta_B$  are considered positive if they direct in the positive  $z$ -direction otherwise they are negative; the end moments  $M_{AB}$  and  $M_{BA}$  and the fixed-end moments  $FEM_{AB}$  and  $FEM_{BA}$  are considered positive if they direct in the positive  $z$ -direction otherwise they are negative; and the sway angle  $\psi_{AB}$  is considered positive if a chord connecting the two end points rotates in the positive  $z$ -direction. Since the local coordinate system is chosen such that the  $z$ -axis directs outward from the paper, the positive  $z$ -direction is therefore equivalent to the counter clockwise direction. For instance, counter clockwise end rotations, counter clockwise sway angles, counter clockwise end moments, and counter clockwise fixed-end moments are considered positive in the present formulation. For the shear forces and axial forces at both ends of the member, they are considered positive if they direct in the positive local  $y$ -axis and positive local  $x$ -axis, respectively.

It should be noted that ones may also be familiar with another choice of local coordinate system shown in Figure 12.3. The local  $z$ -axis is chosen to direct toward the paper while the local  $y$ -axis still directs along the member and the local  $x$ -axis follows the right hand rule. For this particular choice of coordinate, the slope deflection equations remain unchanged except that the clockwise end rotations, the clockwise sway angle, the clockwise end moments, and the clockwise fixed-end moments are considered positive.

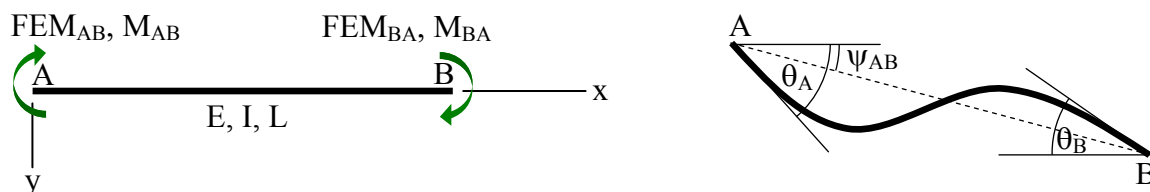
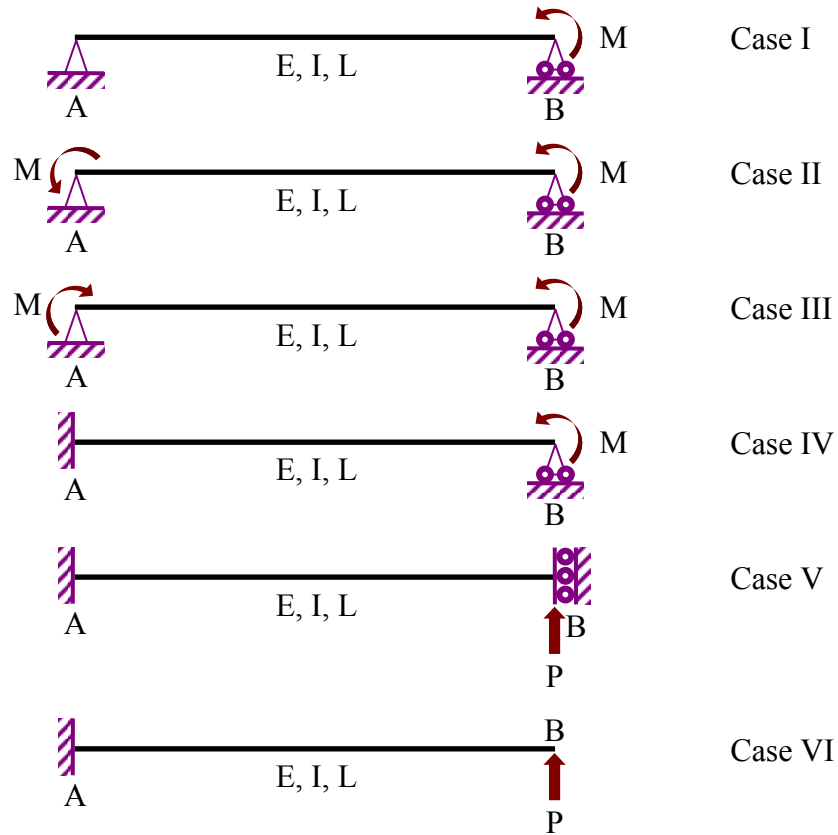


Figure 12.3 Positive sign convention for end rotations, sway angle, end moments and fixed-end moments if the local  $z$ -axis is chosen to direct toward the paper

**Example 12.1** Use the slope-deflection equations to determine the end deflections and end rotations of a single span beam due to applied loads shown below. The length, moment of inertia and Young's modulus of the beam are denoted by  $L$ ,  $I$  and  $E$ , respectively.



**Solution** It should be noted that for all cases considered above, there is no member loads; thus, the fixed-end moments vanish (i.e.  $FEM_{AB} = 0$  and  $FEM_{BA} = 0$ ).

**Case I:** For this particular case, moments at both ends are prescribed and the sway angle vanishes, i.e.  $M_{AB} = 0$ ,  $M_{BA} = M$  and  $\psi_{AB} = 0$ . Substituting this information into the slope-deflection equations leads to

$$M_{AB} = 0 = \frac{2EI}{L}(2\theta_A + \theta_B) \quad \text{and} \quad M_{BA} = M = \frac{2EI}{L}(\theta_A + 2\theta_B)$$

By solving these two linear equations, we obtain the end rotations  $\theta_A = -ML/6EI$  (CW) and  $\theta_B = ML/3EI$  (CCW).

**Case II:** In this case, we have  $M_{AB} = M$ ,  $M_{BA} = M$  and  $\psi_{AB} = 0$ . The slope-deflection equations now become

$$M_{AB} = M = \frac{2EI}{L}(2\theta_A + \theta_B) \quad \text{and} \quad M_{BA} = M = \frac{2EI}{L}(\theta_A + 2\theta_B)$$

Again, by solving above two linear equations, we obtain the end rotations  $\theta_A = \theta_B = ML/6EI$  (CCW).

Case III: In this case, we have  $M_{AB} = -M$ ,  $M_{BA} = M$  and  $\psi_{AB} = 0$ . The slope-deflection equations become

$$M_{AB} = -M = \frac{2EI}{L}(2\theta_A + \theta_B) \quad \text{and} \quad M_{BA} = M = \frac{2EI}{L}(\theta_A + 2\theta_B)$$

By solving above two linear equations, we obtain the end rotations  $\theta_A = -ML/2EI$  (CW) and  $\theta_B = ML/2EI$  (CCW).

Case IV: In this case, we have  $\theta_A = 0$ ,  $M_{BA} = M$  and  $\psi_{AB} = 0$ . The slope-deflection equations become

$$M_{AB} = \frac{2EI}{L}(0 + \theta_B) = \frac{2EI\theta_B}{L} \quad \text{and} \quad M_{BA} = M = \frac{2EI}{L}(0 + 2\theta_B) = \frac{4EI\theta_B}{L}$$

By solving the last equation, it yields  $\theta_B = ML/4EI$  (CCW). Once the rotation at end B is obtained, the moment at end A can readily be obtained from the first slope-deflection equation, i.e.  $M_{AB} = (2EI/L)(ML/4EI) = M/2$ .

Case V: In this case, we have  $\theta_A = 0$  and  $\theta_B = 0$  while both end moments are unknown a priori. However, by considering moment equilibrium of the entire beam, we obtain a relation between  $M_{AB}$ ,  $M_{BA}$  and the applied load P:

$$[\Sigma M_A = 0] \Rightarrow M_{AB} + M_{BA} + PL = 0 \quad (\text{e12.1.1})$$

Also, the slope-deflection equations now become

$$M_{AB} = \frac{2EI}{L}(0 + 0) - \frac{6EI}{L}\psi_{AB} = -\frac{6EI}{L}\psi_{AB} \quad \text{and} \quad M_{BA} = \frac{2EI}{L}(0 + 0) - \frac{6EI}{L}\psi_{AB} = -\frac{6EI}{L}\psi_{AB}$$

By substituting two slope-deflection equations into (e12.1.1) and then solving for the sway angle  $\psi_{AB}$ , this results in  $\psi_{AB} = PL^2/12EI$  (CCW). Since the end A is fully fixed, the deflection at end B can be computed from  $v_B = (\psi_{AB})(L) = PL^3/12EI$  (upward). Once the sway angle  $\psi_{AB}$  is solved, the moments at both ends can be obtained from above slope-deflection equations, i.e.  $M_{AB} = M_{BA} = (-6EI/L)(PL^2/12EI) = -PL/2$  (CW).

Case VI: In this case, we have  $\theta_A = 0$  and  $M_{BA} = 0$  while  $M_{AB}$ ,  $\theta_B$  and  $\psi_{AB}$  are unknown a priori. However, by considering moment equilibrium of the entire beam, we obtain  $M_{AB} = -PL$  (CW). By substituting this information into the slope-deflection equations, we obtain

$$M_{AB} = -PL = \frac{2EI}{L}(0 + \theta_B) - \frac{6EI}{L}\psi_{AB} = \frac{2EI}{L}\theta_B - \frac{6EI}{L}\psi_{AB}$$

$$M_{BA} = 0 = \frac{2EI}{L}(0 + 2\theta_B) - \frac{6EI}{L}\psi_{AB} = \frac{4EI}{L}\theta_B - \frac{6EI}{L}\psi_{AB}$$

By solving above two linear equations, we obtain  $\theta_B = PL^2/2EI$  (CCW) and  $\psi_{AB} = PL^2/3EI$  (CCW). Since the end A is fully fixed, the deflection at end B can be computed from  $v_B = (\psi_{AB})(L) = PL^3/3EI$  (upward).

## 12.3 Computation of Fixed-end Moments

In this section, we further investigate the formula (12.13) and (12.14) and then present an alternative form of such formula convenient for computing the fixed-end moments  $FEM_{AB}$  and  $FEM_{BA}$  for arbitrary member loads. Useful results for fixed-end moments are then summarized in Table 12.1 for certain typical member loads frequently found in idealized structures.

The expressions of the fixed-end moments  $FEM_{AB}$  and  $FEM_{BA}$  given by (12.13) and (12.14) can be re-expressed in a more suitable form as

$$FEM_{AB} = \frac{4}{L} \int_0^L M_o(x) dx - \frac{6}{L^2} \int_0^L x M_o(x) dx \quad (12.17)$$

$$FEM_{BA} = \frac{2}{L} \int_0^L M_o(x) dx - \frac{6}{L^2} \int_0^L x M_o(x) dx \quad (12.18)$$

It is evident that these two expressions involve two identical integrals; the first integral represents the area of the bending moment  $M_o(x)$  over the member AB and the second integral represents the first moment about the end A of the area of the bending moment  $M_o(x)$  over the member AB. To avoid the direct evaluation of such two integrals, the bending moment  $M_o(x)$  is first decomposed into

$$M_o(x) = \sum_{i=1}^N M_{oi}(x) \quad (12.19)$$

where the bending moment  $M_{oi}(x)$  possesses a special form such that its area over the member AB and its centroid can readily be obtained. By inserting the decomposition (12.19) into equations (12.17) and (12.18), it leads to

$$FEM_{AB} = \frac{4}{L} \sum_{i=1}^N A_{oi} - \frac{6}{L^2} \sum_{i=1}^N A_{oi} \bar{x}_{oi} \quad (12.20)$$

$$FEM_{BA} = \frac{2}{L} \sum_{i=1}^N A_{oi} - \frac{6}{L^2} \sum_{i=1}^N A_{oi} \bar{x}_{oi} \quad (12.21)$$

where  $A_{oi}$  is the area of the bending moment diagram  $M_{oi}(x)$  over the member AB and  $\bar{x}_{oi}$  is the distance from the centroid of the bending moment diagram  $M_{oi}(x)$  to the end A.

It should be noted that the key task of computing the fixed-end moments  $FEM_{AB}$  and  $FEM_{BA}$  of the member AB (see Figure 12.4(a)) is to construct the bending moment  $M_o(x)$  (or  $M_{oi}(x)$  for  $i = 1, 2, 3, \dots, N$ ) due to member loads. From the definition in section 12.1, the bending moment  $M_o(x)$  is in fact the bending moment of a simply-supported beam of the same length and subjected to the same set of member loads (see Figure 12.4(b)).

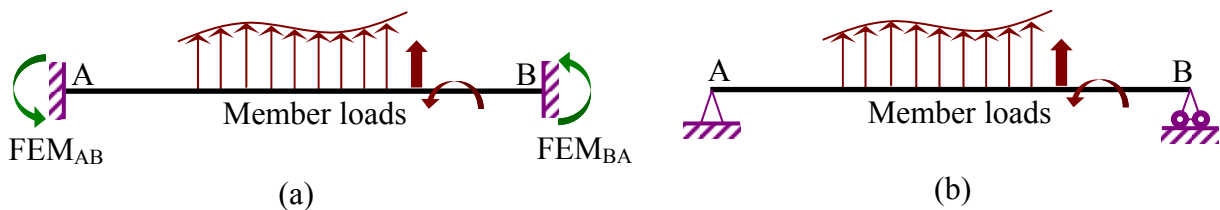
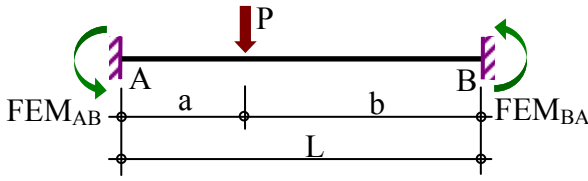
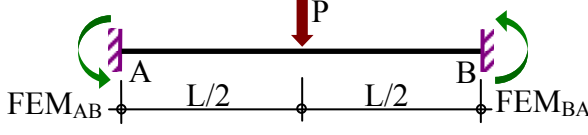
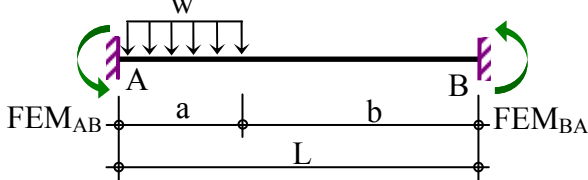
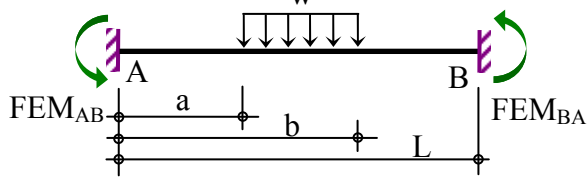
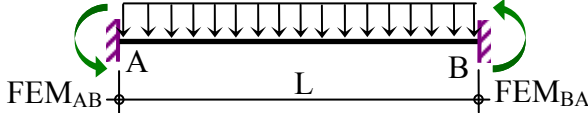
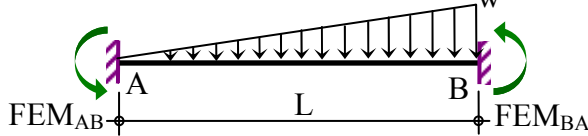
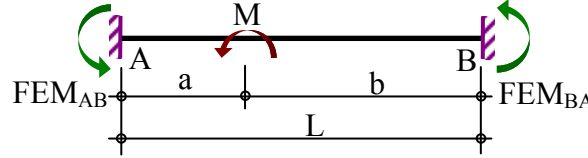
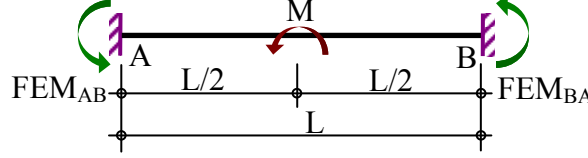


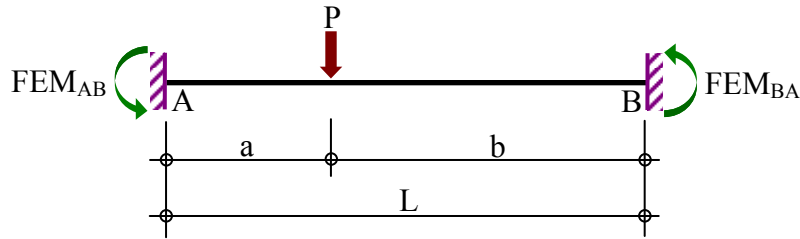
Figure 12.4 (a) Fixed-end structure subjected to member loads and (b) structure for computing  $M_o(x)$  used in the calculation of fixed-end moments  $FEM_{AB}$  and  $FEM_{BA}$



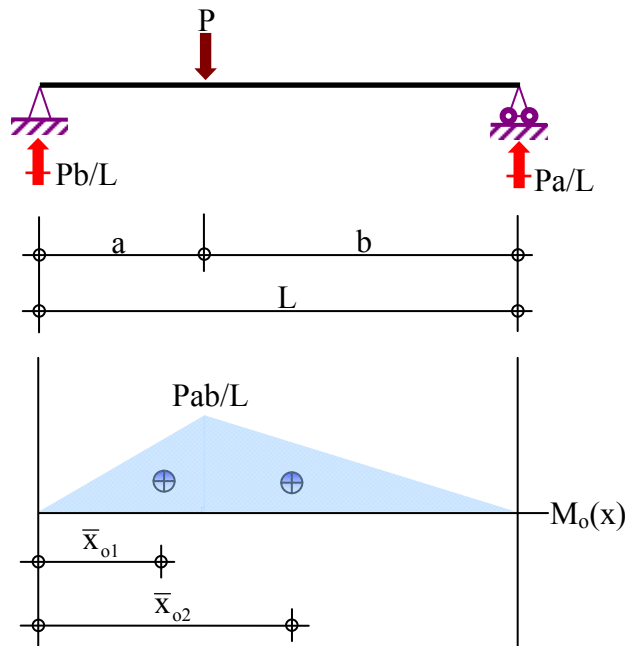
Table 12.1 Fixed-end moments for certain member loads

| Loading conditions                                                                  | $FEM_{AB}$                                                                                                                                                                                                                                            | $FEM_{BA}$                                                                                                                                                                                |
|-------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|    | $\frac{Pab^2}{L^2}$                                                                                                                                                                                                                                   | $-\frac{Pa^2b}{L^2}$                                                                                                                                                                      |
|    | $\frac{PL}{8}$                                                                                                                                                                                                                                        | $-\frac{PL}{8}$                                                                                                                                                                           |
|    | $\frac{wL^2}{12} \left[ 6\left(\frac{a}{L}\right)^2 - 8\left(\frac{a}{L}\right)^3 + 3\left(\frac{a}{L}\right)^4 \right]$                                                                                                                              | $-\frac{wL^2}{12} \left[ 4\left(\frac{a}{L}\right)^3 - 3\left(\frac{a}{L}\right)^4 \right]$                                                                                               |
|   | $\frac{wL^2}{12} \left[ 6\left(\frac{b}{L}\right)^2 - 8\left(\frac{b}{L}\right)^3 + 3\left(\frac{b}{L}\right)^4 \right]$<br>$-\frac{wL^2}{12} \left[ 6\left(\frac{a}{L}\right)^2 - 8\left(\frac{a}{L}\right)^3 + 3\left(\frac{a}{L}\right)^4 \right]$ | $\frac{wL^2}{12} \left[ 4\left(\frac{b}{L}\right)^3 - 3\left(\frac{b}{L}\right)^4 \right]$<br>$-\frac{wL^2}{12} \left[ 4\left(\frac{a}{L}\right)^3 - 3\left(\frac{a}{L}\right)^4 \right]$ |
|  | $\frac{wL^2}{12}$                                                                                                                                                                                                                                     | $-\frac{wL^2}{12}$                                                                                                                                                                        |
|  | $\frac{wL^2}{30}$                                                                                                                                                                                                                                     | $-\frac{wL^2}{20}$                                                                                                                                                                        |
|  | $\frac{Mb(2a-b)}{L^2}$                                                                                                                                                                                                                                | $\frac{Ma(2b-a)}{L^2}$                                                                                                                                                                    |
|  | $\frac{M}{4}$                                                                                                                                                                                                                                         | $\frac{M}{4}$                                                                                                                                                                             |

**Example 12.2** Compute the fixed-end moments of a member subjected to a concentrated force  $P$  with a distance  $a$  from the end A.



**Solution** The bending moment diagram  $M_o(x)$  for this particular member load can readily be constructed by considering a simply-supported beam shown below.



The bending moment diagram  $M_o(x)$  is decomposed into two parts,  $M_{o1}(x)$  and  $M_{o2}(x)$ , where each part forms a triangle as shown above. The area and the distance from their centroid to the end A are given by

$$A_{o1} = \frac{1}{2} \left( \frac{Pab}{L} \right) (a) = \frac{Pa^2b}{2L} \quad ; \quad \bar{x}_{o1} = \frac{2a}{3}$$

$$A_{o2} = \frac{1}{2} \left( \frac{Pab}{L} \right) (b) = \frac{Pab^2}{2L} \quad ; \quad \bar{x}_{o2} = a + \frac{b}{3}$$

From  $A_{o1}$ ,  $A_{o2}$ ,  $\bar{x}_{o1}$ , and  $\bar{x}_{o2}$ , we then obtain

$$\sum_{i=1}^2 A_{oi} = \frac{Pa^2b}{2L} + \frac{Pab^2}{2L} = \frac{Pab}{2}$$

$$\sum_{i=1}^2 A_{oi} \bar{x}_{oi} = \left( \frac{Pa^2b}{2L} \right) \left( \frac{2a}{3} \right) + \left( \frac{Pab^2}{2L} \right) \left( a + \frac{b}{3} \right) = \frac{Pab(2a+b)}{6}$$

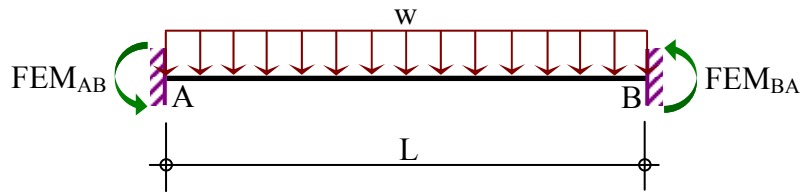
Inserting above results into equations (12.20) and (12.21) yields the fixed-end moments

$$FEM_{AB} = \frac{4}{L} \left( \frac{Pab}{2} \right) - \frac{6}{L^2} \left( \frac{Pab(2a+b)}{6} \right) = \frac{Pab^2}{L^2} \quad ; \quad FEM_{BA} = \frac{2}{L} \left( \frac{Pab}{2} \right) - \frac{6}{L^2} \left( \frac{Pab(2a+b)}{6} \right) = -\frac{Pa^2b}{L^2}$$

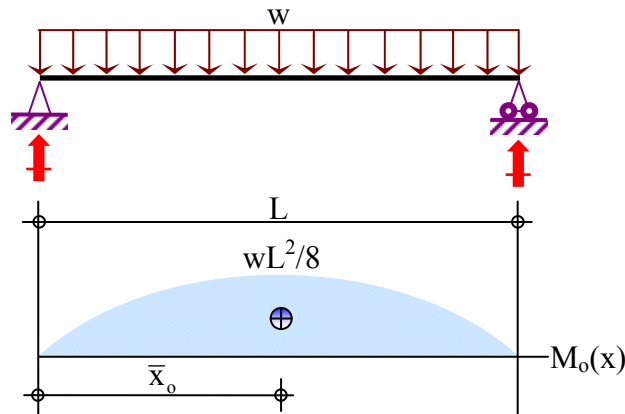
For the special case when the concentrated load  $P$  is applied at the mid span of the member (i.e.  $a = b = L/2$ ), the fixed-end moments reduce to

$$FEM_{AB} = \frac{P(L/2)(L/2)^2}{L^2} = \frac{PL}{8} \quad ; \quad FEM_{BA} = -\frac{P(L/2)^2(L/2)}{L^2} = -\frac{PL}{8}$$

**Example 12.3** Compute the fixed-end moments of a member subjected to a uniformly distributed load  $w$  over its entire span.



**Solution** Similar to the previous example, the bending moment  $M_o(x)$  for this particular case can readily be constructed by considering a simply-supported beam shown below.



Area of the bending moment diagram  $M_o(x)$  and the distance from its centroid to the end A are given by

$$A_o = \frac{2}{3} \left( \frac{wL^2}{8} \right) (L) = \frac{wL^3}{12} \quad ; \quad \bar{x}_o = \frac{L}{2}$$

Substituting  $A_o$  and  $\bar{x}_o$  into (12.20) and (12.21) yields the fixed-end moments

$$FEM_{AB} = \frac{4}{L} \left( \frac{wL^3}{12} \right) - \frac{6}{L^2} \left( \frac{wL^3}{12} \right) \left( \frac{L}{2} \right) = \frac{wL^2}{12} \quad ; \quad FEM_{BA} = \frac{2}{L} \left( \frac{wL^3}{12} \right) - \frac{6}{L^2} \left( \frac{wL^3}{12} \right) \left( \frac{L}{2} \right) = -\frac{wL^2}{12}$$

**Remark:** It is worth noting that the fixed end moments due to any distributed load  $q = q(x)$  can also be computed by using the results of the concentrated load (Example 12.2) instead of using the relations (12.20) and (12.21). To clearly demonstrate the idea, let us treat the distributed load over an infinitesimal element of length  $dx$  (an element connecting the point  $x$  and the point  $x + dx$ ) as a concentrated force  $q(x)dx$  acting to point  $x$ . The fixed end moments due to this infinitesimal concentrated load, denoted by  $dFEM_{AB}$  and  $dFEM_{BA}$ , are given by

$$dFEM_{AB} = \frac{q(x)(L-x)^2 x}{L^2} dx \quad (12.22)$$

$$dFEM_{BA} = -\frac{q(x)(L-x)x^2}{L^2} dx \quad (12.23)$$

Thus, the fixed-end moment due to the distributed load acting on the entire member is obtained by integrating (12.22) and (12.23) from 0 to  $L$ , i.e.

$$FEM_{AB} = \int_0^L \frac{q(x)(L-x)^2 x}{L^2} dx \quad (12.24)$$

$$FEM_{BA} = -\int_0^L \frac{q(x)(L-x)x^2}{L^2} dx \quad (12.25)$$

It should be noted that the distributed load  $q$  in the expressions (12.24) and (12.25) is positive if it directs downward. As an example, let apply the relations (12.24) and (12.25) to re-compute the fixed-end moments due to uniformly distributed load  $q$  over the entire span. This leads to

$$FEM_{AB} = \frac{q}{L^2} \int_0^L (L-x)^2 x dx = \frac{q}{L^2} \left( \frac{L^2 x^2}{2} - \frac{2Lx^3}{3} + \frac{x^4}{4} \right)_0^L = \frac{qL^2}{12} \quad (12.26)$$

$$FEM_{BA} = -\frac{q}{L^2} \int_0^L (L-x)x^2 dx = -\frac{q}{L^2} \left( \frac{Lx^3}{3} - \frac{x^4}{4} \right)_0^L = -\frac{qL^2}{12} \quad (12.27)$$

Similarly, the fixed-end moments due to linearly distributed load  $q(x) = q_0(x/L)$  over the entire span can also be obtained as follows:

$$FEM_{AB} = \frac{q_0}{L^3} \int_0^L (L-x)^2 x^2 dx = \frac{q_0}{L^2} \left( \frac{L^2 x^3}{3} - \frac{2Lx^4}{4} + \frac{x^5}{5} \right)_0^L = \frac{q_0 L^2}{30} \quad (12.28)$$

$$FEM_{BA} = -\frac{q_0}{L^3} \int_0^L (L-x)x^3 dx = -\frac{q_0}{L^2} \left( \frac{Lx^4}{4} - \frac{x^5}{5} \right)_0^L = -\frac{q_0 L^2}{20} \quad (12.29)$$

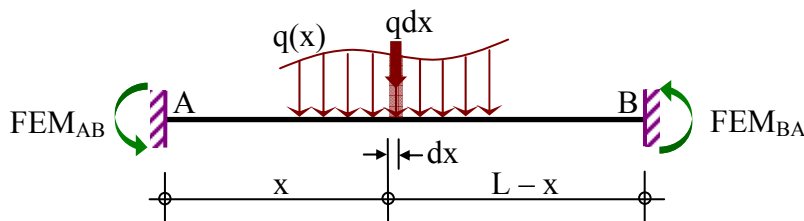
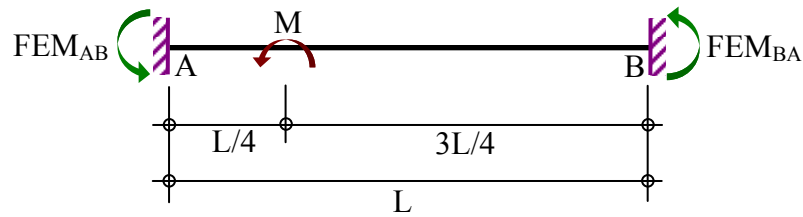
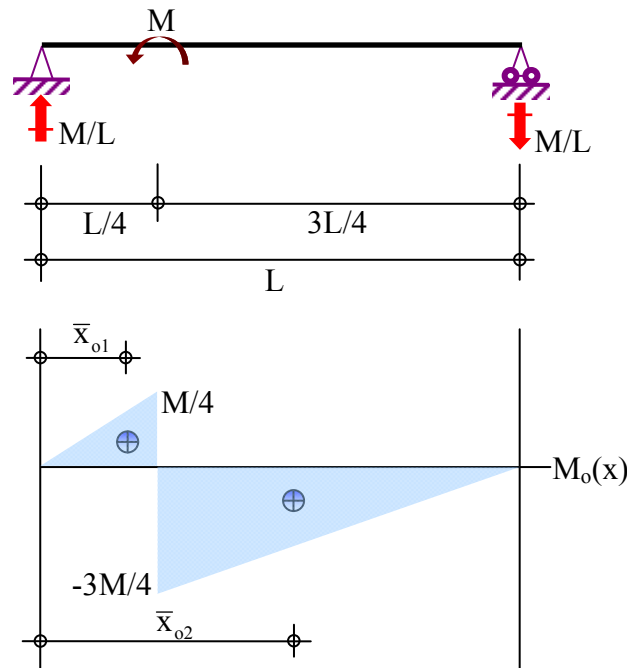


Figure 12.5 Treatment of distributed load over infinitesimal element  $dx$  by concentrated force  $qdx$

**Example 12.4** Compute the fixed-end moments of a member due to a concentrated moment  $M$  acting to its quarter point from the end A.



**Solution** The bending moment diagram  $M_o(x)$  for this particular case is shown below.



This bending moment diagram  $M_o(x)$  is decomposed into two parts,  $M_{o1}(x)$  and  $M_{o2}(x)$ , where each part forms a triangle as shown above. The area and the distance from their centroid to the end A are given by

$$A_{o1} = \frac{1}{2} \left( \frac{M}{4} \right) \left( \frac{L}{4} \right) = \frac{ML}{32} \quad ; \quad \bar{x}_{o1} = \frac{L}{6}$$

$$A_{o2} = \frac{1}{2} \left( -\frac{3M}{4} \right) \left( \frac{3L}{4} \right) = -\frac{9ML}{32} \quad ; \quad \bar{x}_{o2} = \frac{L}{2}$$

From  $A_{o1}$ ,  $A_{o2}$ ,  $\bar{x}_{o1}$ , and  $\bar{x}_{o2}$ , we then obtain

$$\sum_{i=1}^2 A_{oi} = \frac{ML}{32} - \frac{9ML}{32} = -\frac{ML}{4}$$

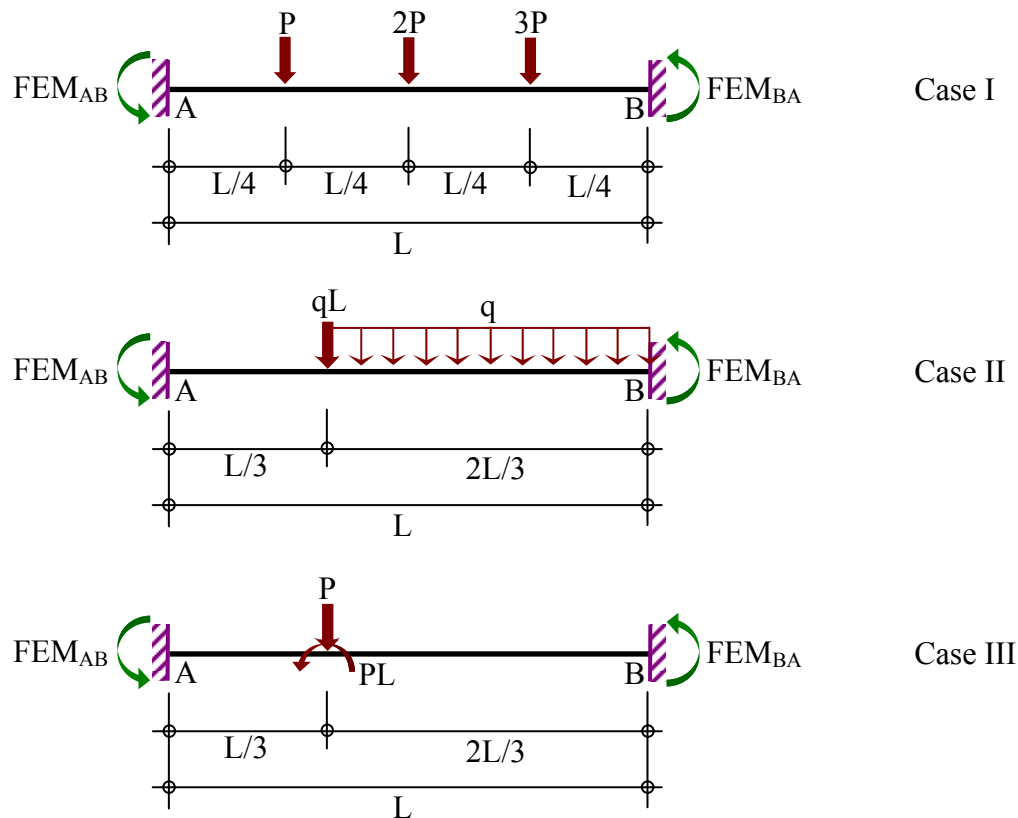
$$\sum_{i=1}^2 A_{oi} \bar{x}_{oi} = \left( \frac{ML}{32} \right) \left( \frac{L}{6} \right) + \left( -\frac{9ML}{32} \right) \left( \frac{L}{2} \right) = -\frac{13ML^2}{96}$$

Inserting above results into equations (12.20) and (12.21) yields the fixed-end moments

$$FEM_{AB} = \frac{4}{L} \left( -\frac{ML}{4} \right) - \frac{6}{L^2} \left( -\frac{13ML^2}{96} \right) = -\frac{3M}{16} \quad ; \quad FEM_{BA} = \frac{2}{L} \left( -\frac{ML}{4} \right) - \frac{6}{L^2} \left( -\frac{13ML^2}{96} \right) = \frac{5M}{16}$$

**Remark:** Since the fixed-end moments are linearly related to the bending moment  $M_0(x)$  and  $M_0(x)$  is clearly a linear function of applied loads, a method of superposition can be applied along with the use of basic results such as those shown in Table 12.1 to compute the fixed-end moments of members subjected to a series of applied loads (see Example 12.5).

**Example 12.5** Use results given in Table 12.1 along with a method of superposition to compute the fixed-end moments of following cases:



**Solution** By using the fixed-end moments in Table 12.1 along with the method of superposition, the fixed-end moments for above three cases are given below.

**Case I:**

$$FEM_{AB} = \frac{P(L/4)(3L/4)^2}{L^2} + \frac{2P(L/2)(L/2)^2}{L^2} + \frac{3P(3L/4)(L/4)^2}{L^2} = \frac{17PL}{32}$$

$$FEM_{BA} = -\frac{P(3L/4)(L/4)^2}{L^2} - \frac{2P(L/2)(L/2)^2}{L^2} - \frac{3P(L/4)(3L/4)^2}{L^2} = -\frac{23PL}{32}$$

**Case II:**

$$FEM_{AB} = \frac{qL(L/3)(2L/3)^2}{L^2} + \frac{qL^2}{12} \left[ 4 \left( \frac{2}{3} \right)^3 - 3 \left( \frac{2}{3} \right)^4 \right] = \frac{16qL^2}{81}$$

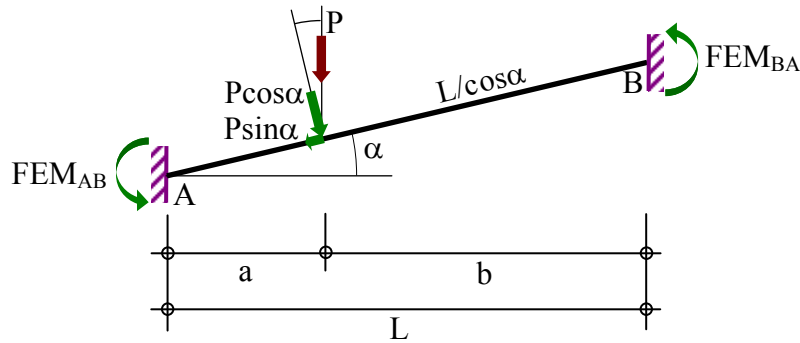
$$FEM_{BA} = -\frac{qL(2L/3)(L/3)^2}{L^2} - \frac{qL^2}{12} \left[ 6\left(\frac{2}{3}\right)^2 - 8\left(\frac{2}{3}\right)^3 + 3\left(\frac{2}{3}\right)^4 \right] = -\frac{4qL^2}{27}$$

**Case III:**

$$FEM_{AB} = \frac{P(L/3)(2L/3)^2}{L^2} + \frac{PL(2L/3)(2L/3 - 2L/3)}{L^2} = \frac{4PL}{27}$$

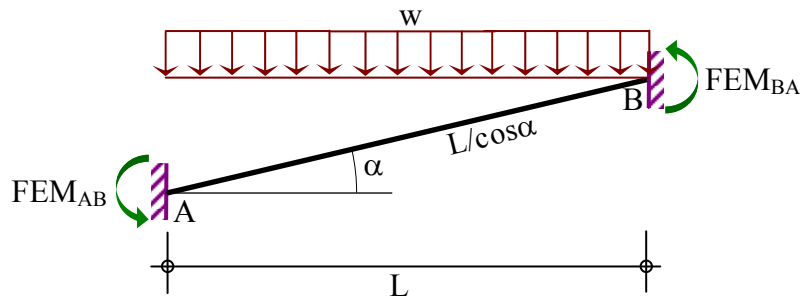
$$FEM_{BA} = -\frac{P(2L/3)(L/3)^2}{L^2} + \frac{PL(L/3)(4L/3 - L/3)}{L^2} = \frac{7PL}{27}$$

**Remark:** Another important remark is associated with the computation of the fixed-end moments of a member subjected to inclined loads (i.e. loads containing both transverse and longitudinal components). By noting that the longitudinal component of member loads produces zero fixed-end moments, the inclined loads can therefore be treated in the same manner as the transverse loads by simply ignoring their longitudinal component. Here, we present two important results that are commonly employed, one associated with an inclined concentrated force and the other corresponding to an inclined, uniformly distributed loads.



$$FEM_{AB} = \frac{(P \cos \alpha)(a/\cos \alpha)(b/\cos \alpha)^2}{(L/\cos \alpha)^2} = \frac{Pab^2}{L^2} \quad (12.30)$$

$$FEM_{BA} = -\frac{(P \cos \alpha)(b/\cos \alpha)(a/\cos \alpha)^2}{(L/\cos \alpha)^2} = -\frac{Pa^2b}{L^2} \quad (12.31)$$



$$FEM_{AB} = \frac{[(wL \cos \alpha) / (L/\cos \alpha)](L/\cos \alpha)^2}{12} = \frac{wL^2}{12} \quad (12.32)$$

$$FEM_{BA} = -\frac{[(wL \cos \alpha) / (L/\cos \alpha)](L/\cos \alpha)^2}{12} = -\frac{wL^2}{12} \quad (12.33)$$

## 12.4 Alternative Form of Slope-deflection Equations

It is evident that the slope-deflection equations (12.15) and (12.16) are not sufficient to solve for the end rotations and sway angle  $\{\theta_A, \theta_B, \psi_{AB}\}$  in terms of the end moments and member loads. This is due primarily to the fact that those three kinematical quantities still contain one mode of rigid body motion (i.e. one associated with the rigid rotation of the member). Now, let us define two new quantities  $\phi_A$  and  $\phi_B$  such that

$$\phi_A = \theta_A - \psi_{AB} \quad (12.34)$$

$$\phi_B = \theta_B - \psi_{AB} \quad (12.35)$$

These two quantities, commonly termed the *relative end rotations*, represent the end rotations of the member measured from a chord connecting both ends of the member in the deformed configuration, see Figure 12.1. The key feature of  $\phi_A$  and  $\phi_B$  is that they completely characterize the deformation of the member and contain no rigid body motion. That is non-zero  $\phi_A$  and  $\phi_B$  always accompany by non-zero bending deformation and vice versa. By substituting equations (12.34) and (12.35) into equations (12.15) and (12.16), it leads to an alternative form of the slope-deflection equations:

$$M_{AB} = \frac{2EI}{L}(2\phi_A + \phi_B) + FEM_{AB} \quad (12.36)$$

$$M_{BA} = \frac{2EI}{L}(\phi_A + 2\phi_B) + FEM_{BA} \quad (12.37)$$

According to the fact that  $\phi_A$  and  $\phi_B$  represent the pure deformation, a system of linear equations (12.36) and (12.37) can be solved to obtain the relative end rotations  $\phi_A$  and  $\phi_B$  in terms of the end moments and the fixed-end moments:

$$\phi_A = \frac{L}{3EI}(M_{AB} - FEM_{AB}) - \frac{L}{6EI}(M_{BA} - FEM_{BA}) \quad (12.38)$$

$$\phi_B = \frac{L}{3EI}(M_{BA} - FEM_{BA}) - \frac{L}{6EI}(M_{AB} - FEM_{AB}) \quad (12.39)$$

Equations (12.36) and (12.37) are useful for obtaining the stiffness information of the flexural member while the flexibility property of the member can be obtained from equations (12.38) and (12.39). For the special that the member is free of member loads, the slope-deflection equations (12.36)-(12.37) and their inverse relations (12.38)-(12.39) simply reduce to

$$M_{AB} = \frac{2EI}{L}(2\phi_A + \phi_B) \quad (12.40)$$

$$M_{BA} = \frac{2EI}{L}(\phi_A + 2\phi_B) \quad (12.41)$$

$$\phi_A = \frac{L}{3EI}M_{AB} - \frac{L}{6EI}M_{BA} \quad (12.42)$$

$$\phi_B = \frac{L}{3EI}M_{BA} - \frac{L}{6EI}M_{AB} \quad (12.43)$$



## 12.5 Slope-deflection Equations for Special Members

The slope-deflection equations (12.15) and (12.16) can be specialized to certain special members, e.g. a member with a prescribed moment at one end, a member with a prescribed shear force at one end, a symmetric member, and an anti-symmetric member. Such specialization can reduce not only the number of the independent slope-deflection equations but also the number of kinematical quantities such as the end rotations and sway angle.

### 12.5.1 Member with prescribed moment at one end

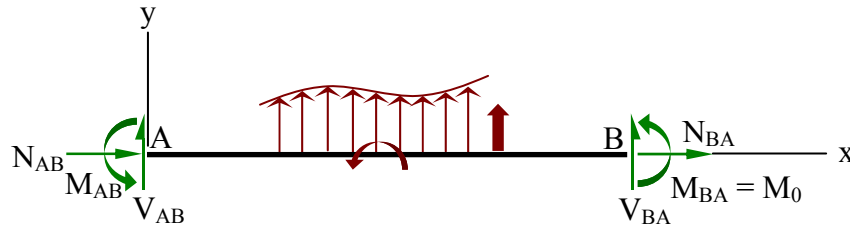


Figure 12.6 A member with prescribed moment at end B

Let us consider first a member AB with a prescribed moment  $M_0$  at the end B (i.e.  $M_{BA} = M_0$ ) as shown in Figure 12.6. By substituting  $M_{BA} = 0$  into (12.16), the end rotation  $\theta_B$  can be solved in terms of other quantities as

$$\theta_B = -\frac{1}{2}\theta_A + \frac{3}{2}\psi_{AB} + \frac{L}{4EI}(M_0 - FEM_{BA}) \quad (12.44)$$

This relation implies that the end rotation  $\theta_B$  is not independent of the end rotation  $\theta_A$  and the sway angle  $\psi_{AB}$ . Once  $\theta_A$  and  $\psi_{AB}$  are known,  $\theta_B$  can readily be computed from (12.44). By replacing  $\theta_B$  from (12.44) into equation (12.15), we obtain a modified slope-deflection equation for the end moment  $M_{AB}$  as

$$M_{AB} = \frac{3EI}{L}\theta_A - \frac{3EI}{L}\psi_{AB} + \overline{FEM}_{AB} \quad (12.45)$$

where  $\overline{FEM}_{AB}$  is a modified fixed-end moment at the end A defined by

$$\overline{FEM}_{AB} = FEM_{AB} - \frac{1}{2}FEM_{BA} + \frac{1}{2}M_0 \quad (12.46)$$

It should be noted that equation (12.45) can be considered as a single slope-deflection equation for this particular member. The condition  $M_{BA} = M_0$  along with the other slope-deflection equation is already used to eliminate the end rotation  $\theta_B$  from the modified equation (12.45). The key advantage gained from using the single modified equation (12.45) instead of the two equations (12.15) and (12.16) is that the number of kinematical unknowns reduces from three to two; only  $\theta_A$  and  $\psi_{AB}$  appear in the modified slope-deflection equation while  $\theta_B$  can be obtained later from (12.44).

For a member with a prescribed moment  $M_0$  at the end A (i.e.  $M_{AB} = M_0$ ), a similar procedure can be used to derive the modified slope-deflection equation for the end moment  $M_{BA}$  and the final result is given by

$$M_{BA} = \frac{3EI}{L}\theta_B - \frac{3EI}{L}\psi_{AB} + \overline{FEM}_{BA} \quad (12.47)$$

where the modified fixed-end moment  $\overline{FEM}_{BA}$  is given by

$$\overline{FEM}_{BA} = FEM_{BA} - \frac{1}{2}FEM_{AB} + \frac{1}{2}M_0 \quad (12.48)$$

The end rotation  $\theta_A$  can then be obtained from

$$\theta_A = -\frac{1}{2}\theta_B + \frac{3}{2}\psi_{AB} + \frac{L}{4EI}(M_0 - FEM_{AB}) \quad (12.49)$$

For the special case that a member contains a hinge or moment release at its end, above results apply by simply replacing  $M_0 = 0$ .

### 12.5.2 Member with a prescribed shear force at one end

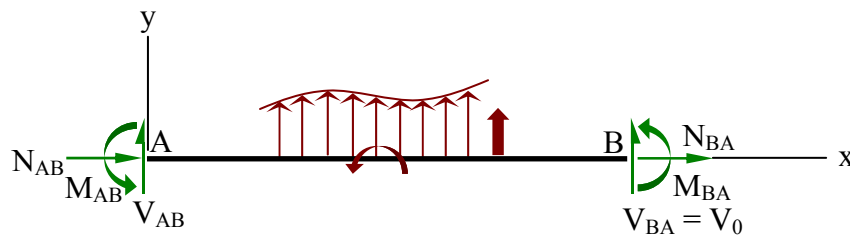


Figure 12.7 A member with prescribed moment at end B

Let us consider a member AB with a prescribed shear force  $V_0$  at the end B (i.e.  $V_{BA} = V_0$ ) as shown in Figure 12.7. By considering moment equilibrium of the member AB about the end A, we obtain

$$V_0L + M_{AB} + M_{BA} + M_{mem} = 0 \Rightarrow M_{AB} + M_{BA} = -V_0L - M_{mem} \quad (12.50)$$

where  $M_{mem}$  is the moment about the end A of all member loads. The relation (12.50) implies that both the end moments  $M_{AB}$  and  $M_{BA}$  are not independent. By substituting the slope-deflection equations (12.15) and (12.16) into (12.50), we can solve for the sway angle in terms of other quantities:

$$\psi_{AB} = \frac{1}{2}(\theta_A + \theta_B) + \frac{(FEM_{AB} + FEM_{BA} + M_{mem} + V_0L)L}{12EI} \quad (12.51)$$

By substituting  $\psi_{AB}$  into equations (12.15) and (12.16), it yields modified slope-deflection equations as

$$M_{AB} = \frac{EI}{L}\theta_A - \frac{EI}{L}\theta_B + \overline{FEM}_{AB} \quad (12.52)$$

$$M_{BA} = \frac{EI}{L}\theta_B - \frac{EI}{L}\theta_A + \overline{FEM}_{BA} \quad (12.53)$$

where  $\overline{FEM}_{AB}$  and  $\overline{FEM}_{BA}$  are modified fixed-end moments defined by

$$\overline{FEM}_{AB} = \frac{1}{2}(FEM_{AB} - FEM_{BA} - M_{mem} - V_0 L) \quad (12.54)$$

$$\overline{FEM}_{BA} = \frac{1}{2}(FEM_{BA} - FEM_{AB} - M_{mem} - V_0 L) \quad (12.55)$$

For this particular case, both the modified slope-deflection equations (12.52) and (12.53) can be used and they involve only two kinematical unknowns (i.e. the end rotations). Once the end rotations are computed, the sway angle can readily be obtained from equation (12.51).

Similarly, the modified slope-deflection equations for a member with a prescribed shear force  $V_0$  at the end A (i.e.  $V_{AB} = V_0$ ) are exactly in the same form as those given by (12.52) and (12.53) except that the modified fixed-end moments  $\overline{FEM}_{AB}$  and  $\overline{FEM}_{BA}$  are defined by

$$\overline{FEM}_{AB} = \frac{1}{2}(FEM_{AB} - FEM_{BA} - M_{mem} + V_0 L) \quad (12.56)$$

$$\overline{FEM}_{BA} = \frac{1}{2}(FEM_{BA} - FEM_{AB} - M_{mem} + V_0 L) \quad (12.57)$$

and the sway angle  $\psi_{AB}$  is given by

$$\psi_{AB} = \frac{1}{2}(\theta_A + \theta_B) + \frac{(FEM_{AB} + FEM_{BA} + M_{mem} - V_0 L)L}{12EI} \quad (12.58)$$

For the special case that a member contains a shear release at its end, above results also apply by simply replacing  $V_0 = 0$ .

### 12.5.3 Symmetric member

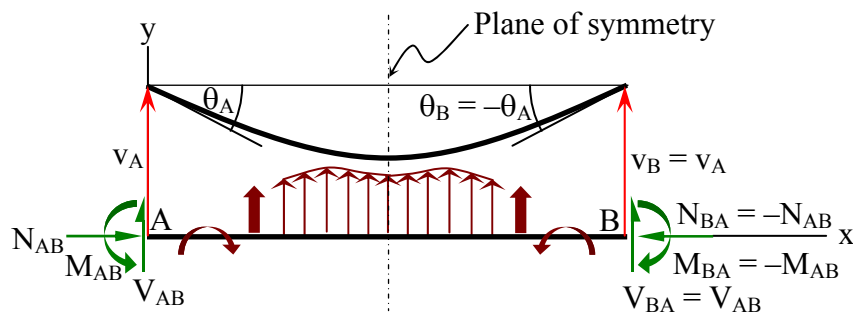


Figure 12.8 Schematic of symmetric member

Let consider a special member such that all quantities (e.g. geometry, member loads, internal forces, deformation, displacement and rotation) are symmetric with respect to a plane normal to the member axis and passing through its center as shown in Figure 12.8. Specifically,  $\theta_B = -\theta_A$ ,  $v_B = v_A$ ,  $u_B = -u_A = 0$ ,  $\psi_{AB} = 0$ ,  $M_{BA} = -M_{AB}$ ,  $V_{BA} = V_{AB}$ ,  $N_{BA} = -N_{AB}$ , and  $FEM_{BA} = -FEM_{AB}$ . By using above symmetric conditions, the slope-deflection equations (12.15) and (12.16) reduce to

$$M_{AB} = \frac{2EI}{L}\theta_A + FEM_{AB} \quad (12.59)$$

$$M_{BA} = \frac{2EI}{L}\theta_B + FEM_{BA} \quad (12.60)$$

The modified slope-deflection equations (12.59) and (12.60) are not independent and one of them may be used in the analysis depending primarily on either one of the end rotations  $\{\theta_A, \theta_B\}$  being chosen as a primary unknown. The key feature of these modified slope-deflection equations is that they contain only one kinematical unknown (i.e. either the end rotation  $\theta_A$  or the end rotation  $\theta_B$ ). Once all quantities at one end are determined, quantities at the other end can readily be obtained from the symmetric conditions.

### 12.5.4 Anti-symmetric member

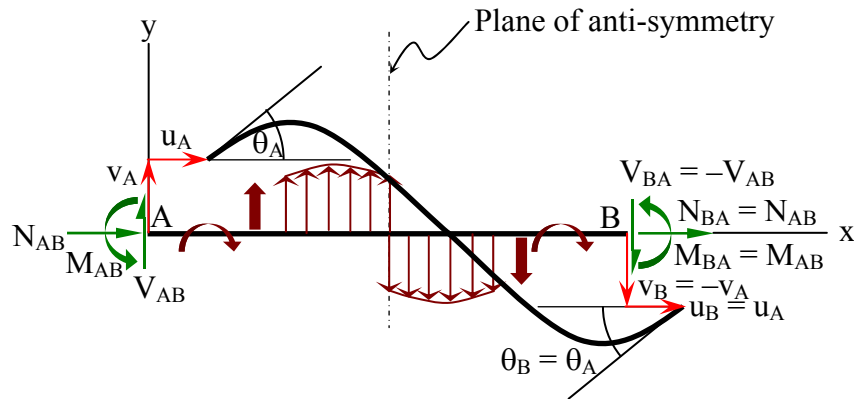


Figure 12.9 Schematic of anti-symmetric member

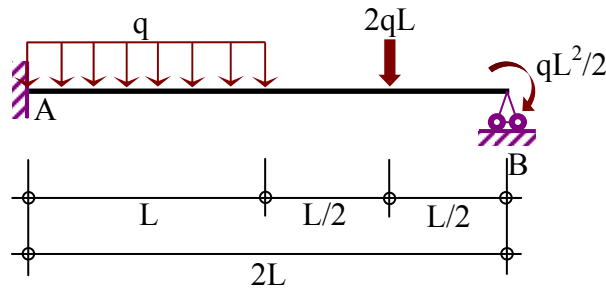
Let consider a special member such that its geometry is symmetric with respect to a plane normal to the member axis and passing through its center while all other quantities (e.g. member loads, internal forces, deformation, displacement and rotation) are anti-symmetric with respect to the same plane as shown in Figure 12.9. Specifically,  $\theta_B = \theta_A$ ,  $v_B = -v_A$ ,  $u_B = u_A$ ,  $M_{BA} = M_{AB}$ ,  $V_{BA} = -V_{AB}$ ,  $N_{BA} = N_{AB}$ , and  $FEM_{BA} = FEM_{AB}$ . By using above anti-symmetric conditions, the slope-deflection equations (12.15) and (12.16) reduce to

$$M_{AB} = \frac{6EI}{L}\theta_A - \frac{6EI}{L}\psi_{AB} + FEM_{AB} \quad (12.61)$$

$$M_{BA} = \frac{6EI}{L}\theta_B - \frac{6EI}{L}\psi_{AB} + FEM_{BA} \quad (12.62)$$

Similar to the symmetric case, the modified slope-deflection equations (12.61) and (12.62) are not independent and only one of them is often used in the analysis. The key feature of these modified slope-deflection equations is that the number of kinematical unknowns reduces from three to two (i.e. they involves only one end rotation and the sway angle). Once all quantities at one end (e.g. rotation, displacement, end moment, shear force, axial force) are solved, quantities at the other end can also be obtained from the anti-symmetric conditions.

**Example 12.6** Use slope-deflection equations to determine the rotation at point B, end moments and end forces, all support reactions, shear force diagram and bending moment diagram of a beam shown below. The flexural rigidity EI is constant throughout.



**Solution** Since the flexural rigidity EI is constant throughout, the entire beam is treated as a single member AB. The fixed-end moments  $FEM_{AB}$  and  $FEM_{BA}$  can readily be obtained from Table 12.1 along with the superposition as follows:

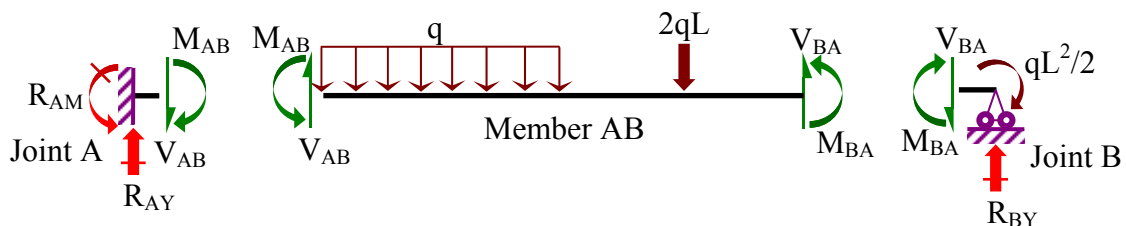
$$FEM_{AB} = \frac{(2qL)(3L/2)(L/2)^2}{(2L)^2} + \frac{q(2L)^2}{12} \left[ 6\left(\frac{1}{2}\right)^2 - 8\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^4 \right] = \frac{5qL^2}{12}$$

$$FEM_{BA} = -\frac{(2qL)(3L/2)^2(L/2)}{(2L)^2} - \frac{q(2L)^2}{12} \left[ 4\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^4 \right] = -\frac{2qL^2}{3}$$

Since the end rotation  $\theta_A$  and the sway angle  $\psi_{AB}$  vanish, the slope-deflection equations for the member AB become

$$M_{AB} = \frac{2EI}{(2L)}(2(0) + \theta_B) - \frac{6EI}{(2L)}(0) + \frac{5qL^2}{12} = \frac{EI\theta_B}{L} + \frac{5qL^2}{12} \quad (\text{e12.6.1})$$

$$M_{BA} = \frac{2EI}{(2L)}(0 + 2\theta_B) - \frac{6EI}{(2L)}(0) - \frac{2qL^2}{3} = \frac{2EI\theta_B}{L} - \frac{2qL^2}{3} \quad (\text{e12.6.2})$$



To determine the unknown rotation  $\theta_B$ , we first write the moment equilibrium at joint B as follows:

$$[\Sigma M_{\text{joint B}} = 0] \Rightarrow M_{BA} + \frac{qL^2}{2} = 0 \quad (\text{e12.6.3})$$

By substituting  $M_{BA}$  from (e12.6.2) into (e12.6.3) and then solving for  $\theta_B$ , it leads to

$$\frac{2EI\theta_B}{L} - \frac{2qL^2}{3} + \frac{qL^2}{2} = 0 \Rightarrow \theta_B = \frac{qL^3}{12EI} \quad \text{CCW}$$

Once the end rotation  $\theta_B$  is solved, the end moment  $M_{AB}$  can be obtained from (e12.6.1):

$$M_{AB} = \frac{EI}{L} \left( \frac{qL^3}{12EI} \right) + \frac{5qL^2}{12} = \frac{qL^2}{2} \quad \text{CCW}$$

By considering equilibrium of the member AB and using  $M_{AB} = qL^2/2$  and  $M_{BA} = -qL^2/2$ , the end shear forces  $V_{AB}$  and  $V_{BA}$  can be obtained as follows:

$$[\Sigma M_{\text{end A}} = 0] \Rightarrow qL^2/2 - qL^2/2 + V_{BA}(2L) - (qL)(L/2) - (2qL)(3L/2) = 0 \Rightarrow V_{BA} = \frac{7qL}{4}$$

$$[\Sigma F_{Y, \text{member AB}} = 0] \Rightarrow V_{AB} + V_{BA} - qL - 2qL = 0 \Rightarrow V_{AB} = \frac{5qL}{4}$$

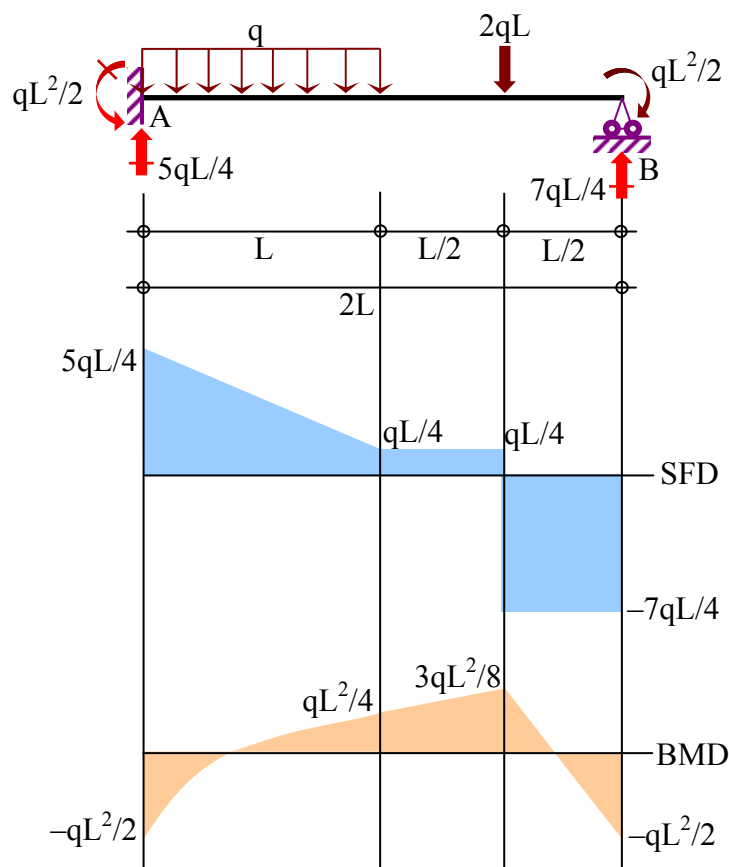
All support reactions can readily be obtained from equilibrium of joints A and B as shown below:

$$[\Sigma M_{\text{joint A}} = 0] \Rightarrow R_{AM} = M_{AB} = \frac{qL^2}{2} \quad \text{CCW}$$

$$[\Sigma F_{Y, \text{joint A}} = 0] \Rightarrow R_{AY} = V_{AB} = \frac{5qL}{4} \quad \text{Upward}$$

$$[\Sigma F_{Y, \text{joint B}} = 0] \Rightarrow R_{BY} = V_{BA} = \frac{7qL}{4} \quad \text{Upward}$$

The shear force and bending moment diagrams are shown below.



## 12.6 Analysis of Structures by Slope-deflection Equations

In this section, we clearly demonstrate the application of the slope-deflection equations (12.15) and (12.16) to the analysis of beams and plane frames. The procedure begins with the discretization of a given structure into a collection of straight, prismatic members and the identification of all independent, kinematical unknowns (i.e. nodal rotations and sway displacements) resulting from such discretization. The slope-deflection equations are subsequently applied to each member to express the end moments in terms of the end rotations and the sway angle. A set of governing equations for the entire structure is established by enforcing equilibrium of moments at all nodes and equilibrium of forces associated with all sway modes. The final step involves solving this set of linear algebraic equations. Once all kinematical unknowns are determined, other quantities of interest (e.g. end moments and forces, support reactions, SFD and BMD) can readily be computed from the slope-deflection equations along with static equilibrium equations. Each of these steps is clearly described below.

### 12.6.1 Discretization and identification of kinematical unknowns

Let us first discretize a given structure into  $N_m$  members with  $N_n$  nodes. For instance, a beam shown in Figure 12.10(a) is discretized into 3 members (i.e.  $N_m = 3$ ; members AB, BC, and CD) with 4 nodes (i.e.  $N_n = 4$ ; nodes A, B, C, and D) and a gable frame shown in Figure 12.10(b) is discretized into 4 members (i.e.  $N_m = 4$ ; members AB, BC, CD, and DE) with 5 nodes (i.e.  $N_n = 5$ ; nodes A, B, C, D, E). The key criterion employed in such discretization is that the slope-deflection equations (12.15) and (12.16) can apply to all discretized members. More specifically, each member must have constant flexural rigidity  $EI$ , contain exactly two nodes at its ends, contain no interior support, and contain no interior internal release (e.g. hinge and shear release). It is worth noting that the discretization pattern of any given structure is not unique but depends primarily on the collection of members chosen. For instance, if a point E of the beam shown in Figure 12.10(a) is also chosen as a node, the discretized structure therefore consists of 4 members with 5 nodes. While the discretization of the structure can be carried out in an arbitrary manner and with a matter of preference, it is common to minimize the number of nodes and members in order to minimize the number of corresponding kinematical unknowns. This will be apparent in the discussion below.

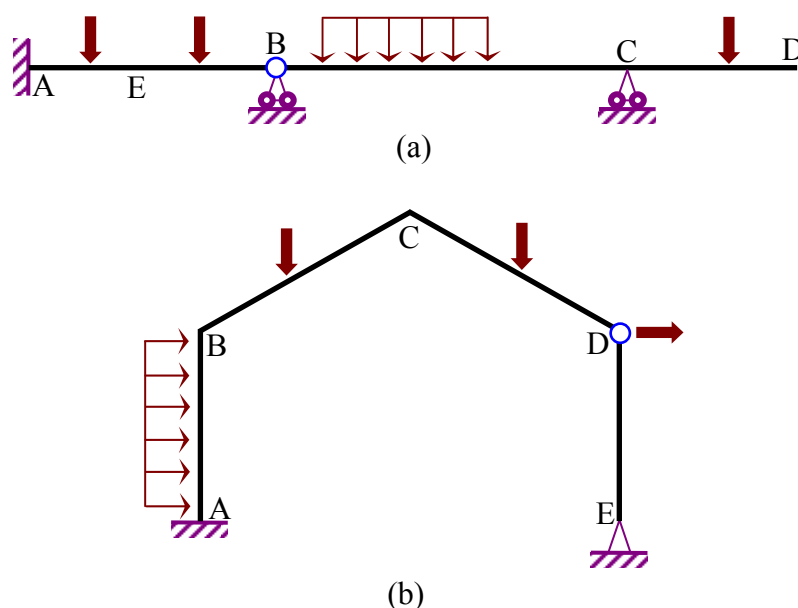


Figure 12.10: Discretization of (a) continuous beam and (b) gable frame into collection of members

In the analysis by slope-deflection equations, the primary unknowns involve two main types of kinematical quantities, *rotational degrees of freedom* and *sway degrees of freedom*. The former, also termed the *nodal rotation*, denotes the rotation at all nodes while the latter, sometimes called the *sway displacement*, denotes the quantity that represents the translational movement of nodes and, in turn, produces the sway angle to each member. Once the given structure is discretized or, equivalently, nodes and members are chosen, all primary unknowns can be identified. It should be noted that the number of such primary unknowns depends primarily on the discretization pattern.

The number of rotational degrees of freedom, denoted by  $N_r$ , is related to the number of nodes ( $N_n$ ), the number of moment releases or hinges ( $N_h$ ) and the number of rotational constraints provided by supports ( $N_{cr}$ ) via a simple relation

$$N_r = N_n + N_h - N_{cr} \quad (12.63)$$

For instance, if the beam shown in Figure 12.10(a) is discretized into three members (i.e. members AB, BC, and CD) with four nodes (i.e. nodes A, B, C, and D), it will contain four rotational degrees of freedom  $\theta_{BL}$ ,  $\theta_{BR}$ ,  $\theta_C$ , and  $\theta_D$  (i.e.  $N_n = 4$ ,  $N_h = 1$ ,  $N_{cr} = 1 \Rightarrow N_r = 4 + 1 - 1 = 4$ ). The rotation at node A is known a priori (i.e.  $\theta_A = 0$ ) and is not treated as an unknown and, due to the presence of a hinge at node B, the rotation at a point just to the left and a point just to the right of node B are, in general, different and must be treated as two unknowns. If this beam is discretized into four members (i.e. members AE, EB, BC, and CD) with five nodes (i.e. nodes A, E, B, C, and D), it will contain five rotational degrees of freedom  $\theta_E$ ,  $\theta_{BL}$ ,  $\theta_{BR}$ ,  $\theta_C$ ,  $\theta_D$  (i.e.  $N_n = 5$ ,  $N_h = 1$ ,  $N_{cr} = 1 \Rightarrow N_r = 5 + 1 - 1 = 5$ ). Similarly if the gable frame shown in Figure 12.10(b) is discretized into four members (i.e. members AB, BC, CD, and DE) with five nodes (i.e. nodes A, B, C, D, and E), it will contain five rotational degrees of freedom  $\theta_B$ ,  $\theta_C$ ,  $\theta_{DL}$ ,  $\theta_{DR}$ ,  $\theta_E$  (i.e.  $N_n = 5$ ,  $N_h = 1$ ,  $N_{cr} = 1 \Rightarrow N_r = 5 + 1 - 1 = 5$ ).

The number of independent sway degrees of freedom of a discretized structure, denoted by  $N_s$ , can be obtained by investigating the freedom of nodes to undergo the translational movement. The key factors that affect the number of sway degrees of freedom are the number of nodes, the translational constraints provided by supports, the internal or deformation constraints (e.g. member inextensibility), and the configuration and arrangement of members within the structure. For beams, the number of independent sway degrees of freedom is given by

$$N_s = N_n - N_{cv} \quad (12.64)$$

where  $N_{cv}$  is the number of translational constraints in the transverse direction of a beam provided by all supports. It should be noted that the inextensibility and small rotation assumptions allow the longitudinal displacement at any point of a statically stable beam be discarded. For instance, the beam shown in Figure 12.10(a) has only one sway degree of freedom (i.e.  $N_n = 4$ ,  $N_{cv} = 3 \Rightarrow N_s = 4 - 3 = 1$ ) for the previous discretization into 3 members and 4 nodes. The deflection at point D, denoted by  $\Delta_D$ , can be chosen to represent such sway degree of freedom. The displacement of the beam associated with the sway degree of freedom is indicated by a red line in Figure 12.11(a). It will become evident later that in order to obtain complete information about the sway angle of any member, it is sufficient to represent the actual sway displacement (the red line) only by a simpler schematic consisting of straight lines or chords connecting between member ends in the deformed configuration termed the *sway pattern* and indicated by a dash line in Figure 12.11(a). If this beam is discretized into 4 members with 5 nodes as considered previously, it will contain two sway degrees of freedom, one associated with the deflection at point D and the other associated with the deflection at point E. The actual sway displacement and the corresponding sway pattern are shown in Figure 12.11(b) by the red line and dash line, respectively.



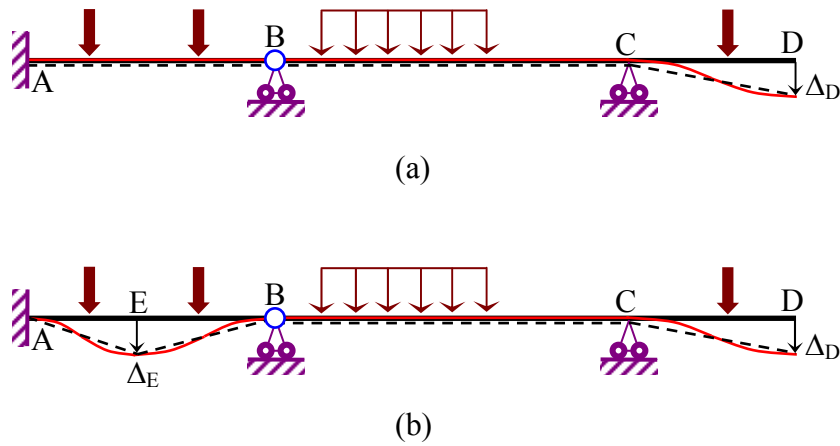


Figure 12.11: Actual sway displacement and sway pattern of a discretized beam consisting of (a) 4 nodes and 3 members and (b) 5 nodes and 4 members

For plane rigid frames, the number of independent sway degrees of freedom can be computed from

$$N_s = 2N_n - N_{ct} - N_{cd} \quad (12.65)$$

where  $N_{ct}$  is the total number of translational constraints provided by all supports and  $N_{cd}$  is the total number of independent internal constraints introduced by the member inextensibility. It should be emphasized that the member inextensibility poses certain restrictions on the movement of all nodes; in particular, all nodes cannot displace in an independent fashion but they must maintain the original member length in the deformed configuration. For a small displacement and rotation assumption, the member length is considered preserved if and only if the projection of a deformed member onto its initial axis possesses the same length as that of the undeformed member. Equivalently, the longitudinal displacements at both ends of a member are identical, i.e.  $u_A = u_B$  as shown schematically in Figure 12.12.

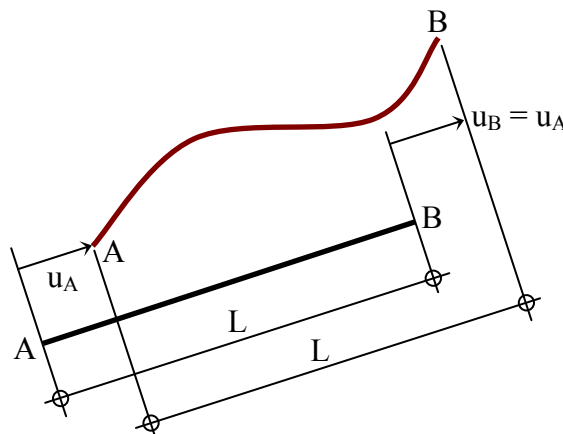


Figure 12.12: Schematic of undeformed and deformed configurations of inextensible member

For instance, the gable frame discretized as shown in Figure 12.10(b) contains only two independent sway degrees of freedom (i.e.  $N_n = 5$ ,  $N_{ct} = 4$ ,  $N_{cd} = 4 \Rightarrow N_s = 2(5) - 4 - 4 = 2$ ). The

first sway degree of freedom, denoted by the horizontal displacement  $\Delta_B$  of point B, corresponds to side-sway of the frame without the movement of point D whereas the second degree of freedom, denoted by the horizontal displacement  $\Delta_D$  of point D, corresponds to side-sway of the frame without the movement of point B. It is clear that these two degrees of freedom are independent. The actual sway displacements and the sway patterns associated with those two sway degrees of freedom are illustrated in Figure 12.13. It is important to note that a choice of sway degrees of freedom is not unique and generally a matter of preference. The key point is to ensure that all sway degrees of freedom are included and they are independent.

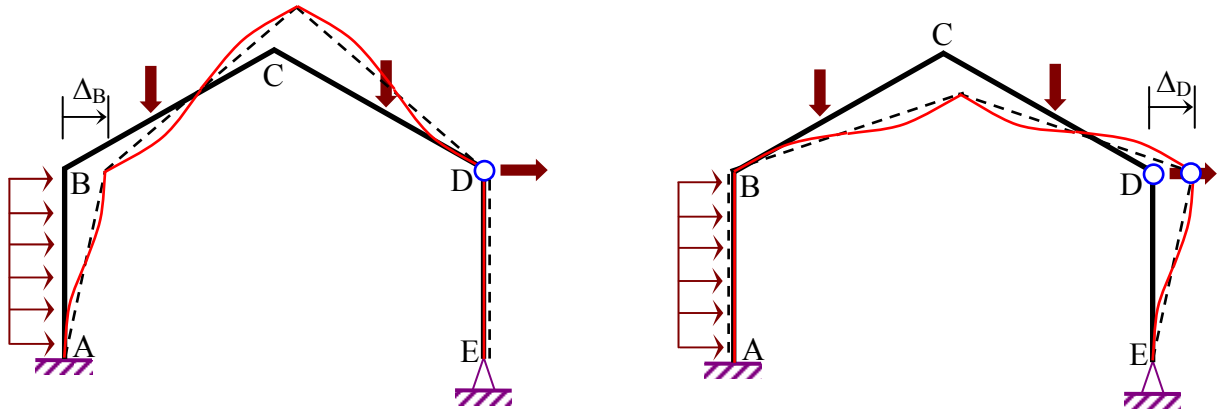


Figure 12.13: Schematic of actual sway displacements and sway patterns associated with two sway degrees of freedom of discretized gable frame shown in Figure 12.10

According to the inextensibility assumption, one additional constraint is imposed for each member; therefore, the number of independent translational degrees of freedom per one two-dimensional member reduces from four to three. Such three independent degrees of freedom include the transverse displacements at both ends and the longitudinal displacement at one end; the longitudinal displacement at the other end must be the same in order to maintain the member length. By using this observation as a guideline, the number of independent internal constraints  $N_{cd}$  cannot exceed the number of members  $N_m$  (i.e.  $N_{cd} \leq N_m$ ). For various structural configurations (see Figure 12.10 for examples), the equality holds (i.e.  $N_{cd} = N_m$ ). However, there are certain configurations where the strong inequality holds (i.e.  $N_{cd} < N_m$ ). This situation occurs when internal constraints provided by many members are not all independent. For example, a plane frame shown in Figure 12.14 has  $N_n = 6$ ,  $N_m = 5$ ,  $N_{ct} = 7$ , and  $N_{cd} = 4$ , and therefore possesses only one sway degree of freedom (i.e.  $N_s = 2(6) - 7 - 4 = 1$ ). The inextensibility of members AB and BD along with the translational constraints provided by supports A and D is sufficient to fully prevent the translation of point B. The inextensibility of a member BF therefore provides no additional constraint affecting the translation of the point B.

Now, the total number of primary (kinematical) unknowns of a discretized structure, denoted by  $N_f$ , is equal to the sum of the number of rotational degrees of freedom and the number of sway degrees of freedom, i.e.

$$N_f = N_r + N_s \quad (12.66)$$

The discretized structure with  $N_s = 0$  is generally known as a *non-sway* structure while the discretized structure with  $N_s > 0$  is termed a *sway* structure. It is important to emphasize that based on this definition a given structure can be either a sway or non-sway structure depending on the discretization.

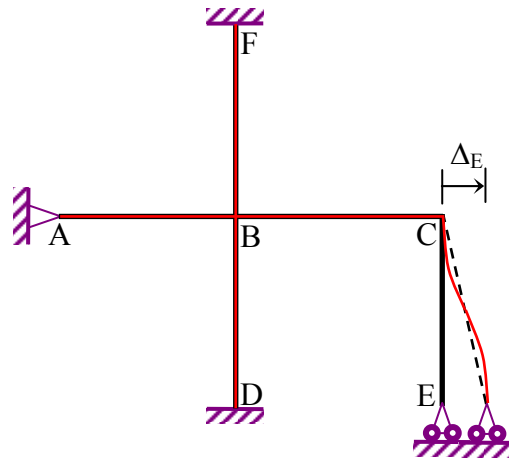
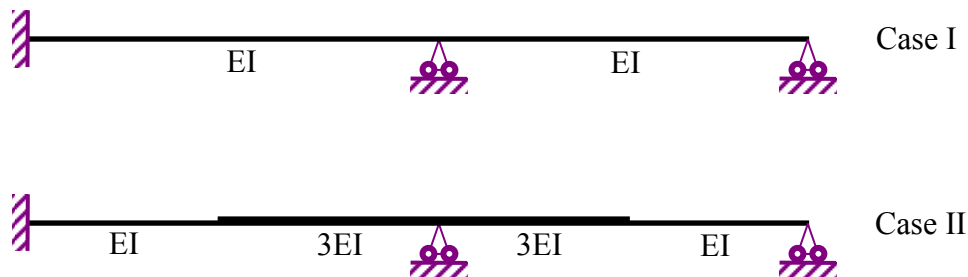
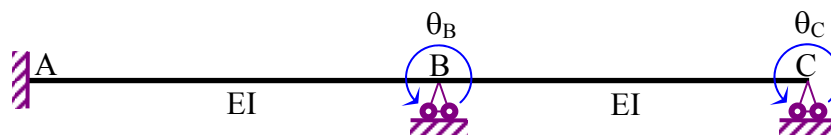


Figure 12.14: Schematic of plane frame with the number of independent internal constraints  $N_{cd}$  less than the number of members  $N_m$ .

**Example 12.7** Discretize two continuous beams shown below by minimizing the number of members as many as possible and then determine the total number of primary (kinematical) unknowns. If the discretized beam is a sway structure, sketch all independent sway patterns.

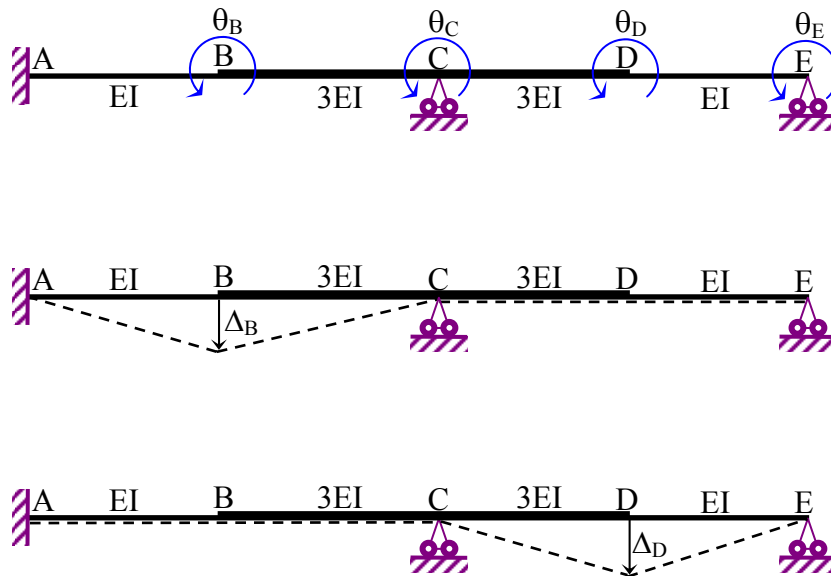


**Solution** Since the flexural rigidity  $EI$  of a beam in case I is constant throughout, it can then be discretized into two members AB and BC with three nodes A, B and C as shown below. For the discretized beam, we have  $N_m = 2$ ,  $N_n = 3$ ,  $N_h = 0$ ,  $N_{cr} = 1$  and  $N_{cv} = 3$ . The number of rotational degrees of freedom, the number of sway degrees of freedom, and the number of primary unknowns are obtained from equations (12.63), (12.64) and (12.66) as follows:  $N_r = 3 + 0 - 1 = 2$ ,  $N_s = 3 - 3 = 0$ , and  $N_f = 2 + 0 = 2$ . Two nodal rotations, denoted by  $\theta_B$  and  $\theta_C$ , are shown below and since  $N_s = 0$ , the discretized beam is a non-sway structure.

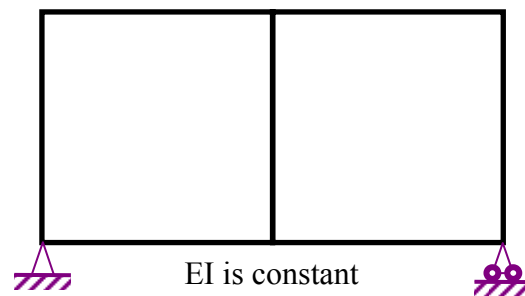


For case II, at least four members must be used in the discretization due to the non-uniform flexural rigidity. A particular discretized beam with four members (AB, BC, CD, and DE) and five nodes (A, B, C, D, and E) is shown below. For this case, we have  $N_m = 4$ ,  $N_n = 5$ ,  $N_h = 0$ ,  $N_{cr} = 1$  and  $N_{cv} = 3$ . The number of rotational degrees of freedom, the number of sway degrees of freedom, and the number of primary unknowns are obtained in the same fashion as follows:  $N_r = 5 + 0 - 1 = 4$ ,  $N_s = 5$

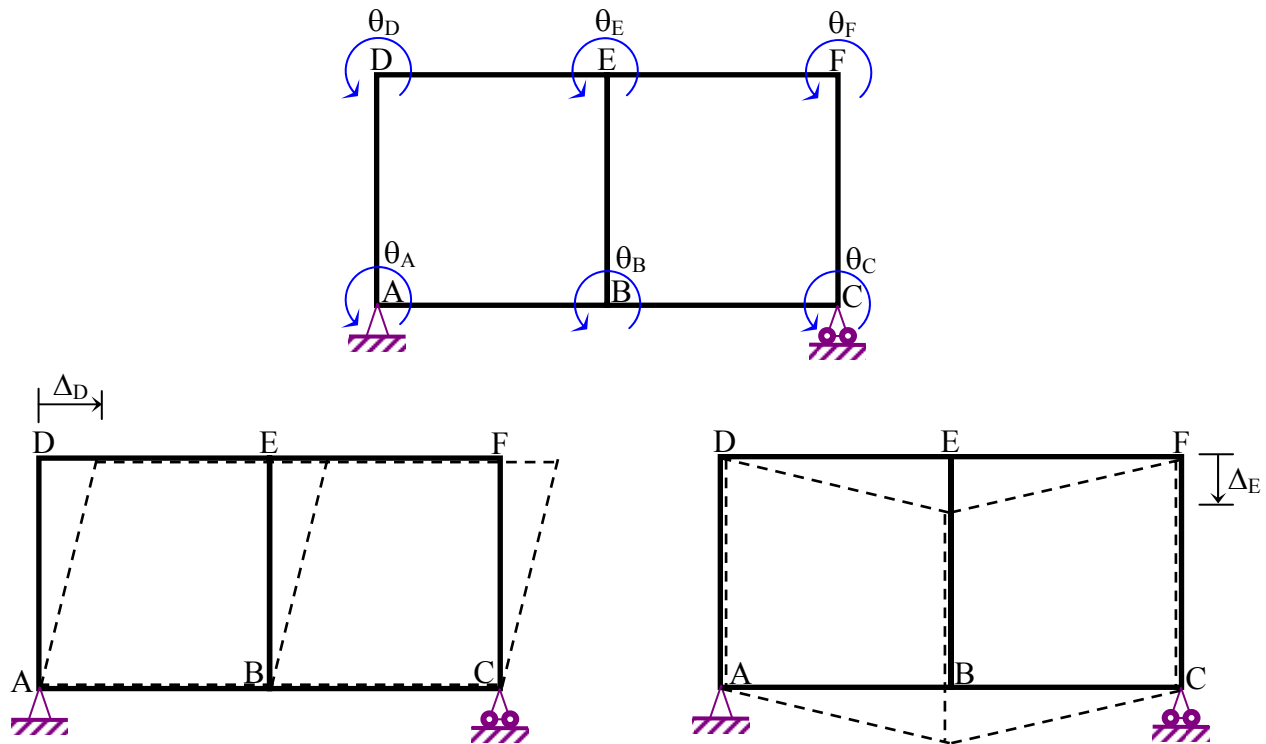
$-3 = 2$ , and  $N_f = 4 + 2 = 6$ . Four nodal rotations, denoted by  $\theta_B$ ,  $\theta_C$ ,  $\theta_D$ , and  $\theta_E$ , are shown below and since  $N_s = 2 > 0$ , the discretized beam is a sway structure. The first sway degree of freedom is chosen to correspond to the deflection of point D whereas the second sway degree of freedom is chosen to be associated with the deflection of point E. Two sway patterns associated with these two independent sway degrees of freedom are shown below.



**Example 12.8** Discretize a frame shown below by minimizing the number of members as many as possible and then determine the total number of primary (kinematical) unknowns. If the discretized frame is a sway structure, sketch all independent sway patterns.



**Solution** From the configuration and constant flexural rigidity of a given frame, at least seven members must be used in the discretization. The discretized frame consisting of seven members (AB, BC, DE, EF, AD, BE, and CF) and six nodes (A, B, C, D, E, and F) is shown below. For such discretization, we have  $N_m = 7$ ,  $N_n = 6$ ,  $N_h = 0$ ,  $N_{cr} = 0$ ,  $N_{ct} = 3$ , and  $N_{cd} = 7$ . The number of rotational degrees of freedom, the number of sway degrees of freedom, and the number of primary unknowns are obtained from equations (12.63), (12.65) and (12.66) as follows:  $N_r = 6 + 0 - 0 = 6$ ,  $N_s = 2(6) - 3 - 7 = 2$ , and  $N_f = 6 + 2 = 8$ . Six nodal rotations, denoted by  $\theta_A$ ,  $\theta_B$ ,  $\theta_C$ ,  $\theta_D$ ,  $\theta_E$ , and  $\theta_F$ , are shown below and since  $N_s = 2 > 0$ , the discretized frame is a sway structure. The first sway degree of freedom is chosen to correspond to the horizontal displacement of point D whereas the second sway degree of freedom is chosen to be associated with the vertical displacement of point E. Two sway patterns associated with these two independent sway degrees of freedom are shown below.



## 12.6.2 End rotations and sway angle of members

After the given structure is discretized and all primary (kinematical) unknowns are defined, the end rotations and sway angle of all members can readily be expressed in terms of those unknowns. By exploiting the compatibility of the rotation at all nodes, it is obvious that the end rotation of the member must be equal to the rotational degree of freedom at the node connected to that member end. For instance, the end rotations of all members in the discretized frame shown in Example 12.8 can be obtained in terms of six rotational degrees of freedom  $\{\theta_A, \theta_B, \theta_C, \theta_D, \theta_E, \theta_F\}$  as shown in Table 12.2.

Table 12.2 End rotations of all members of discretized frame in Example 12.8

| Member | Rotation at end 1 | Rotation at end 2 |
|--------|-------------------|-------------------|
| AB     | $\theta_A$        | $\theta_B$        |
| BC     | $\theta_B$        | $\theta_C$        |
| DE     | $\theta_D$        | $\theta_E$        |
| EF     | $\theta_E$        | $\theta_F$        |
| AD     | $\theta_A$        | $\theta_D$        |
| BE     | $\theta_B$        | $\theta_E$        |
| CF     | $\theta_C$        | $\theta_F$        |

For a sway discretized structure, the sway angle of each member is generally non-zero and can be expressed in terms of the sway degrees of freedom. This step is nontrivial and requires geometric consideration of the structure under the side-sway along with the length constraints posed by the inextensibility assumption. An ingredient that is found very useful and helpful to achieve this task is a sketch of the sway patterns. The translational movements of all nodes in the sway pattern must occur in a manner that ensures the preservation of the member length. In particular, for each

sway degree of freedom, the displacement at all nodes can completely be determined in terms of that degree of freedom and the sway angle of each member can subsequently be computed from the transverse component of the displacement at both ends via the relation (12.12). Finally, the total sway angle of each member can readily be obtained by summing all the sway angles resulting from each sway degree of freedom.

To clearly demonstrate the steps explained above, let us consider a frame with geometry and flexural rigidity shown in Figure 12.15(a). This frame is first discretized into 3 members (AB, BC, and CD) with 4 nodes (A, B, C, and D). The number of sway degrees of freedom of this discretized structure is obviously equal to 2 (i.e.  $N_n = 4$ ,  $N_{ct} = 3$ ,  $N_{cd} = 3 \Rightarrow N_s = 2(4) - 3 - 3 = 2$ ). The first sway degree of freedom, with its sway pattern shown in Figure 12.15(b), corresponds to the side-sway of the frame without the movement of point D. Let us choose the horizontal displacement at point B, denoted by  $\Delta_1$ , to represent the first sway degree of freedom. The translation of all other nodes can then be expressed in terms of  $\Delta_1$  as follows. By employing the length preservation of members AB and CD, the vertical displacements of points B and C essentially vanish and, by enforcing the length preservation of a member BC, the horizontal displacement of point C must be equal to  $\Delta_1$ . With this data, the sway angles of all members produced by the first sway degree of freedom can be computed and results are reported in Table 12.3. For the second sway degree of freedom, we choose the side-sway of the frame such that the movement of point D does not vanish and the corresponding sway pattern is shown in Figure 12.15(c). By choosing the horizontal displacement at point D, denoted by  $\Delta_2$ , to represent this sway degree of freedom, the sway angles of all members in terms of this sway degree of freedom are also given in Table 12.3. It should be noted that these two sway degrees of freedom completely describes the side-sway movement of the discretized structure. Finally, the total sway angle of each member can readily be obtained by summing the sway angles obtained from the two cases (see results in Table 12.3).

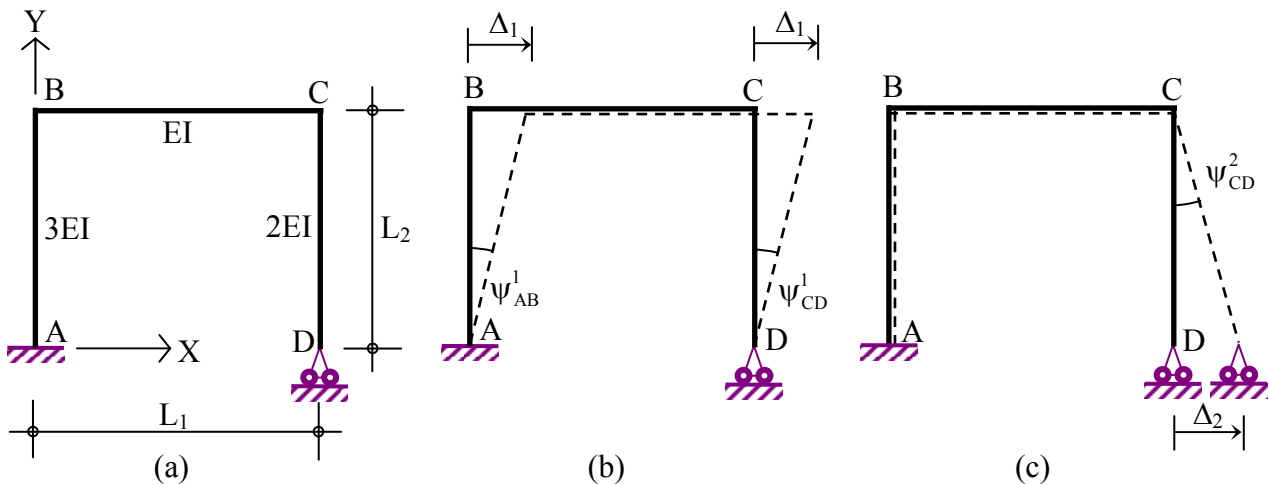


Figure 12.15: (a) Schematic of discretized frame containing 3 members and 4 nodes, (b) sway pattern associated with 1<sup>st</sup> sway DOF, and (c) sway pattern associated with 2<sup>nd</sup> sway DOF

Table 12.3 Sway angle of all members of discretized frame shown in Figure 12.15(a)

| Member | Sway angle due to 1 <sup>st</sup> sway DOF | Sway angle due to 2 <sup>nd</sup> sway DOF | Total sway angle            |
|--------|--------------------------------------------|--------------------------------------------|-----------------------------|
| AB     | $-\Delta_1/L_2$                            | 0                                          | $-\Delta_1/L_2$             |
| BC     | 0                                          | 0                                          | 0                           |
| CD     | $-\Delta_1/L_2$                            | $\Delta_2/L_2$                             | $(\Delta_2 - \Delta_1)/L_2$ |

Consider next a plane frame with geometry and flexural rigidity shown in Figure 12.16(a). This frame is first discretized into 3 members (AB, BC and CD) with 4 nodes (A, B, C and D). The number of sway degrees of freedom of this discretized structure is equal to 1 (i.e.  $N_n = 4$ ,  $N_{ct} = 4$ ,  $N_{cd} = 3 \Rightarrow N_s = 2(4) - 4 - 3 = 1$ ). Let us choose the horizontal displacement at point B, denoted by  $\Delta_1$ , to represent this single sway degree of freedom and let the corresponding sway pattern be shown in Figure 12.16(b). Since a point A is fully fixed, a point B can only displace in the direction perpendicular to the member AB in order to maintain the length of a member AB. Thus, the vertical displacement of point B must be equal to  $\Delta_1$  (since the member AB is oriented 45 degree from the X-axis) and the total displacement of point B is equal to  $\sqrt{2}\Delta_1$ . Due to the length constraint of a member CD along with the constraint provided by a support at point D, the vertical displacement of point C must vanish and, due to the length constraint of a member BC, the horizontal displacement of point C must be equal to  $\Delta_1$ . Once the displacements at all nodes are expressed in terms of  $\Delta_1$ , the sway angles of all members can readily be computed using the relation (12.12) and results are reported in Table 12.4.

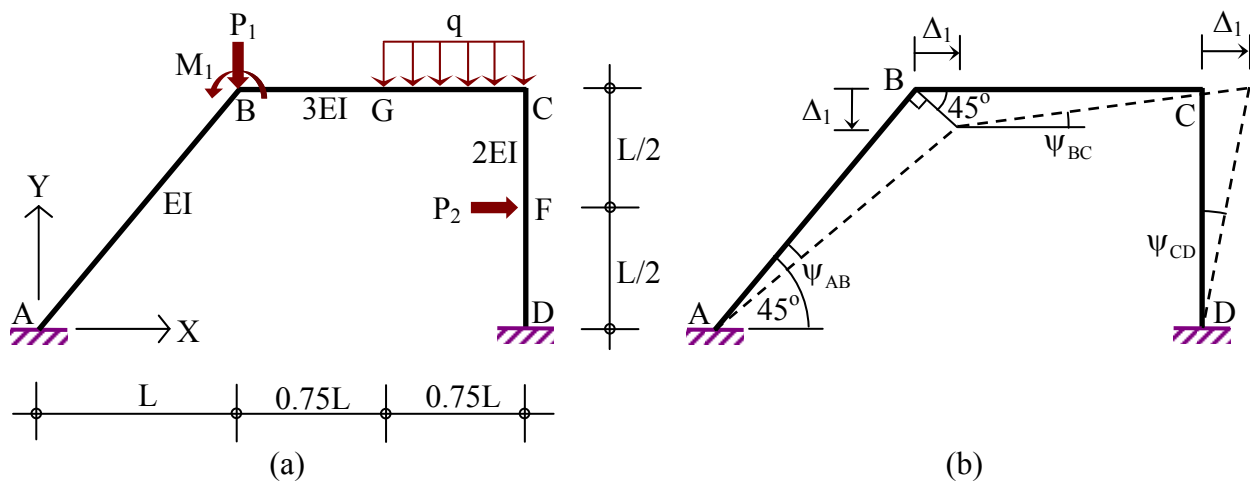


Figure 12.16: (a) Schematic of discretized frame containing 3 members and 4 nodes and (b) sway pattern associated with the sway degree of freedom.

Table 12.4 Sway angles of all members of discretized frame shown in Figure 12.16(a)

| Member | Length      | Sway angle     |
|--------|-------------|----------------|
| AB     | $\sqrt{2}L$ | $-\Delta_1/L$  |
| BC     | $1.5L$      | $2\Delta_1/3L$ |
| CD     | $L$         | $-\Delta_1/L$  |

### 12.6.2.1 Determination of sway angle by instantaneous center of rotations (ICR)

It is evident that determination of the sway angles of all members in the sway structure by the direct geometric consideration and constraints introduced by the length preservation can become nontrivial and requires substantial effort when the configuration of the structure is relatively complex, for instance, structures consisting of multiple and inclined members. Here, we introduce an alternative to express the sway angles in terms of sway degrees of freedom by using the concept of an *instantaneous center of rotation* (ICR).

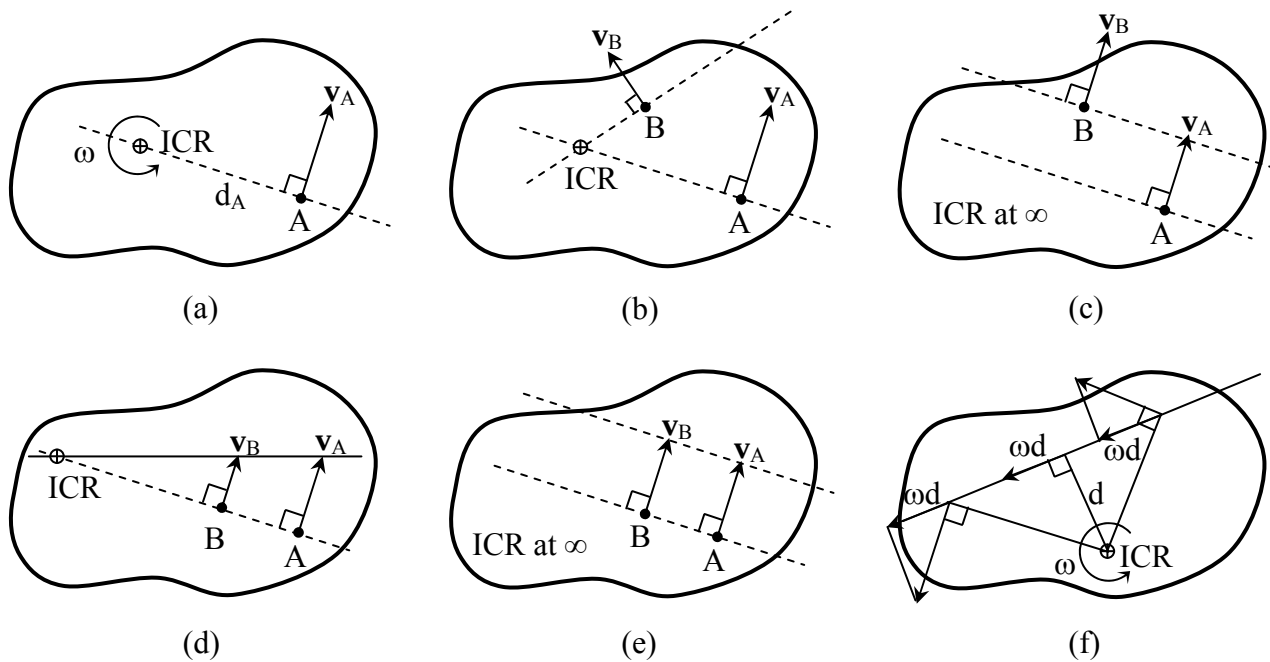


Figure 12.17: Schematic of rigid body in motion

To demonstrate the basic concept, let us consider a plane *rigid body* in motion as shown in Figure 12.17. At any instant during the motion, there exists a point that acts as a center of rotation of the entire rigid body and this particular point is known as the *instantaneous center of rotation* or *ICR* of the body. The key properties of the ICR can be summarized as follows.

- The magnitude of the velocity at any point of the body is proportional to the distance from the ICR and the direction of the velocity is perpendicular to a line connecting the ICR and that particular point. Let  $\omega$  be an angular velocity of the body at a particular instant, the velocity at point A is given by

$$\mathbf{v}_A = \omega d_A \mathbf{n}_A \quad (12.67)$$

where  $d_A$  is the distance from the ICR to the point A and  $\mathbf{n}_A$  is a unit vector normal to a line connecting the ICR and the point A. It is evident from (12.67) that the velocity at the ICR vanishes.

- A fixed point within the body (if exists) is always the ICR.
- The ICR of a body undergoing a pure translation is at infinity.
- If the direction of the velocity is known at two different points, there are three possibilities about the ICR. First, if two known directions are coincident and not perpendicular to a line connecting the two points, the body undergoes a pure translation in the direction of the known direction and the ICR is therefore at infinity (see Figure 12.17(c)). Second, if the two known directions are coincident and perpendicular to a line connecting the two points, the ICR can be either at infinity (see Figure 12.17(e)) or on a line passing through the two points but its exact location cannot be determined except the magnitudes of the velocity at those two points are also known (see Figure 12.17(d)). Finally, if two known directions are different, the ICR can be obtained uniquely from the intersection of two lines that pass through each point and perpendicular to the known direction of the velocity at that point (see Figure 12.17(b)).
- At any instant, if the ICR and the magnitude of the velocity at one particular point are known, the angular velocity can be determined from the relation (12.67).



- The velocities at any point along the same straight line always have the same components along that straight line (see Figure 12.17(f)).

The ICR concept has a direct application in the determination of sway angles of all members in terms of sway degrees of freedom since only the sway pattern not the actual sway displacement, as evident from previous discussion, is required in such task. To sketch a sway pattern associated with a given sway degree of freedom, we simply imagine that all members in the discretized structure are fully rigid (to meet the length preservation condition) and connected only by pinned joints (only chord rotation is needed). Once a particular motion is introduced to this fictitious structure, the ICR of all rigid members is obtained first by using information such as support constraints and known directions of nodal movements. Velocity at any point and angular velocity of any member can subsequently be determined from the ICR concept. By multiplying the resulting velocity and angular velocity by an infinitesimal time, it finally yields the infinitesimal displacement at each point and the sway angle of each member. By setting the displacement component at one particular node to represent the sway degree of freedom, the displacement at all other nodes and the sway angles of all members can completely be expressed in terms of such sway degree of freedom.

Based on above idea, we can summarize essential steps using the ICR concept to express the sway angles of all members in terms of a particular sway degree of freedom as follows:

- (1) Choose a displacement component at a particular node to represent a sway DOF
- (2) Determine the ICR of certain members using support conditions and known directions of nodal movements;
- (3) Compute the sway angle of members that both the ICR and one displacement component at its end are known by using the following relation

$$\psi = \Delta / d \quad (12.68)$$

where  $\Delta$  is the sway angle,  $\Delta$  is the known displacement component and  $d$  is the shortest distance from the ICR to the direction of known displacement component as indicated in Figure 12.18. It is important to emphasize that (12.68) gives only the magnitude of the sway angle whereas its direction (CW or CCW) should be obvious from the direction of  $\Delta$  and the location of the ICR. For instance, the sway angles of the left and right members shown in Figure 12.18 are in CW and CCW directions, respectively.

- (4) Use information from members whose sway angle is known to identify ICR and one displacement component at the end of remaining members and then determine their sway angle; and
- (5) Repeat step (4) until the sway angles of all members are obtained.

For a discretized structure containing multiple sway degrees of freedom, above procedure can be repeated until all sway degrees of freedom are considered. The total sway angle of any member is equal to the sum of sway angles resulting from all sway degrees of freedom.

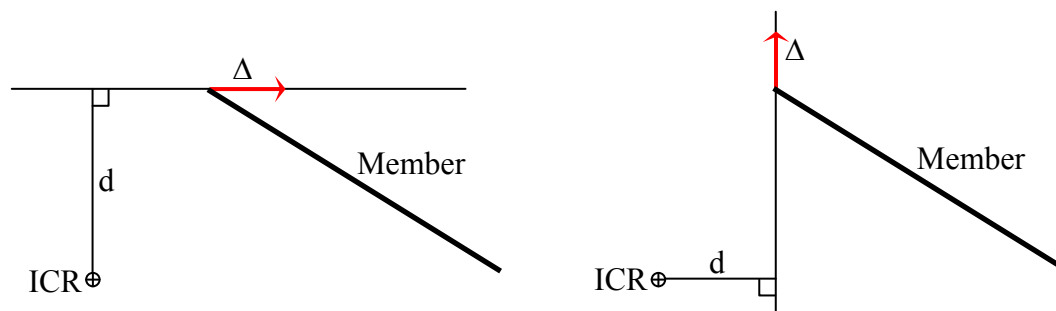


Figure 12.18: Schematic of members whose ICR and one displacement component are known

To clearly demonstrate above steps, let us consider again the frame shown in Figure 12.16. First, we choose the horizontal displacement at point B, denoted by  $\Delta_1$ , to represent the sway degree of freedom. Since a point A is fully fixed, it is therefore the ICR of the member AB. By using the relation (12.68), the sway angle of the member AB is equal to  $\psi_{AB} = -\Delta_1/L$  (the negative sign indicates that the sway angle is in CW direction). By using the fact that a point A is the ICR of the member AB and a point D is the ICR of the member CD (i.e. point D is also a fixed point), the displacement of point B must be perpendicular to a line AB and the displacement of point C must be perpendicular to a line CD. With this information, the ICR of the member BC can readily be determined and it is simply the intersection of the line AB and the line CD, denoted by point E as shown in Figure 12.19. Again, by using the relation (12.68), the sway angle of a member BC is equal to  $\psi_{BC} = \Delta_1 / \overline{CE} = 2\Delta_1 / 3L$ . From the known ICR and sway angle of the member BC, the displacement of point C can be computed as follows  $\Delta_C = \psi_{BC} \overline{CE} = (2\Delta_1 / 3L)(1.5L) = \Delta_1$ . Finally, with known  $\Delta_C$  and the ICR at point D, the sway angle of the member CD can be obtained from (12.68) as  $\psi_{CD} = -\Delta_C / \overline{CD} = -\Delta_1 / L$ .

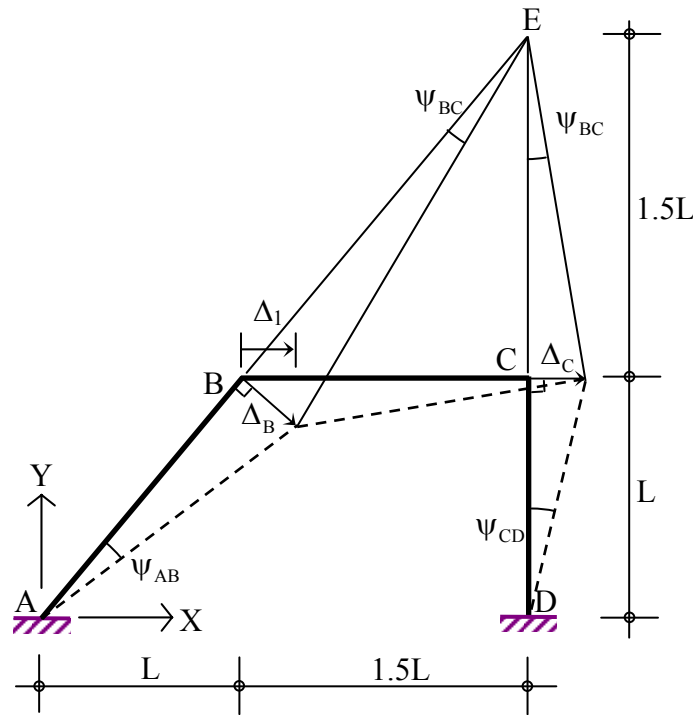


Figure 12.19: Schematic of the sway pattern of frame shown in Figure 12.16 by ICR concept

### 12.6.3 Determination of member fixed-end moments

The fixed-end moment at both ends of any member can readily be computed using the relations (12.17)-(12.18) or (12.20)-(12.21) provided that member loads are given. For certain types of member loads, values of the fixed-end moments can readily be obtained by using results given in Table 12.1 along with the method of superposition.

### 12.6.4 Slope-deflection equations in terms of primary unknowns

The slope-deflection equations for each member in terms of primary unknowns and member loads can readily be obtained by substituting the end rotations in terms of rotational degrees of freedom,

the sway angle in terms of the sway degrees of freedom, and the fixed-end moments into equations (12.15) and (12.16).

## 12.6.5 Set up equilibrium equations

In this step, we establish a framework to set up a set of independent equations sufficient for solving all primary unknowns (i.e. nodal rotations and sway displacements). By recalling the use of three basic sets of equations (i.e. equilibrium equations, kinematics, and constitutive relations), the constitutive relations have already been utilized in the derivation of the slope-deflection equations, kinematics have already been employed both in the member level (to derive the slope-deflection equations) and in the structure level (to relate the primary unknowns to the end rotations and the sway angle of all members) while equilibrium equations have already been used only in the member level to derive the slope-deflection equations. It still remains to enforce static equilibrium in the structural level to relate end moments and end forces of all members to nodal loads. These global equilibrium equations form a set of equations governing all primary unknowns.

The key governing equations can be separated into two sets: the first set containing  $N_r$  equations associated with nodes whose rotations are treated as the rotational degrees of freedom whereas the second set containing  $N_s$  equilibrium equations associated with all sway degrees of freedom. The total number of equilibrium equations is equal to  $N_r + N_s$  which is identical to the number of primary unknowns. It should be remarked that for a non-sway structure, only the first set is required to solve for all primary unknowns (i.e. rotational degrees of freedom).

### 12.6.5.1 Moment equilibrium at nodes

By considering a free body diagram of a particular node (whose rotation is one of primary unknowns), the moment equilibrium of that node can readily be enforced and yields one equation in terms of end moments of members joining that nodes and the external moment applied directly to that node (if exists). As an example, let us consider the frame shown in Figure 12.16(a). It is recalled that this structure is discretized into three members with four nodes and, therefore, it consists of two rotational degrees of freedom denoted by  $\theta_B$  and  $\theta_C$ . For this particular case, two moment equilibrium equations, one associated with moment equilibrium of node B and the other associated with moment equilibrium of node C, must be set up. To construct these two equilibrium equations, we first sketch free body diagrams of node B and node C as shown in Figure 12.20 and then enforce equilibrium of moments at these two nodes. The resulting equations are given by

$$[\Sigma M_{\text{node B}} = 0] \Rightarrow M_{BA} + M_{BC} - M_1 = 0$$

$$[\Sigma M_{\text{node C}} = 0] \Rightarrow M_{CB} + M_{CD} = 0$$

where  $M_1$  is the counter-clockwise moment applied to node B and  $M_{BA}$ ,  $M_{BC}$ ,  $M_{CB}$ , and  $M_{CD}$  are positive end moments transferred from the member ends to nodes. It is important to emphasize that due to the positive sign convention for end moments considered in the derivation of slope-deflection equations, positive end moments transferred to nodes possess clockwise directions.

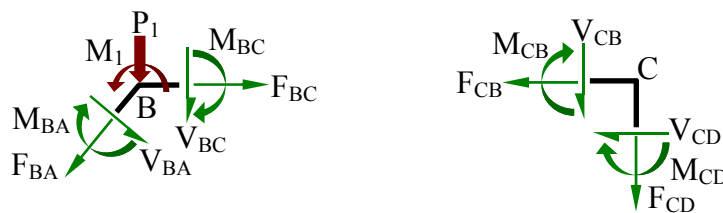


Figure 12.20: Free body diagram of nodes B and C of frame shown in Figure 12.16(a)

In general, moment equilibrium at any node P takes the form

$$\Sigma M_{PQ} - M_{P0} = 0 \quad (12.69)$$

where  $M_{P0}$  is a counter-clockwise moment applied to node P ( $M_{P0}$  is considered negative if it directs in the clockwise direction),  $M_{PQ}$  is the end moment transferred from a member PQ to node P, and the summation is taken over all members joining the node P.

### 12.6.5.2 Equilibrium equations associated with sway degrees of freedom

To establish  $N_s$  independent equilibrium equations associated with all  $N_s$  sway degrees of freedom, the principle of virtual work has been found very attractive over a conventional procedure based upon a method of section and free body diagrams, and will be employed here. The principle of virtual work states that *the structure is in equilibrium if and only if the internal virtual work (or the virtual strain energy) and the external virtual work are identical for all admissible virtual displacements*. Since structures under consideration are always in equilibrium with external applied loads, the internal virtual work and the external virtual work must be the same for any chosen virtual displacement

To construct an equilibrium equation associated with the  $i^{\text{th}}$  sway degree of freedom, we choose a special virtual displacement identical to the  $i^{\text{th}}$  sway pattern (see Figure 12.21). For this particular choice of the virtual displacement, all members are straight and subjected only to rigid rotation (equal to their sway angle) while rotations at all nodes vanish. The member deformation is therefore localized only at their ends due to the side-sway (see Figure 12.21(b)). As a result, the internal virtual work of a generic element PQ due to the localized, relative rotation at both ends, denoted by  $\delta W_{\text{int}}^{PQ}$ , is given by

$$\delta W_{\text{int}}^{PQ} = -(M_{PQ} + M_{QP})\psi_{PQ} \quad (12.70)$$

where  $\psi_{PQ}$  is the sway angle of the member PQ and  $M_{PQ}$  and  $M_{QP}$  are end moments of the member PQ. The internal virtual work of the entire structure, denoted by  $\delta W_{\text{int}}$ , is equal to the sum of the internal virtual work produced by all members, i.e.

$$\delta W_{\text{int}} = \Sigma W_{\text{int}}^{PQ} = -\Sigma (M_{PQ} + M_{QP})\psi_{PQ} \quad (12.71)$$

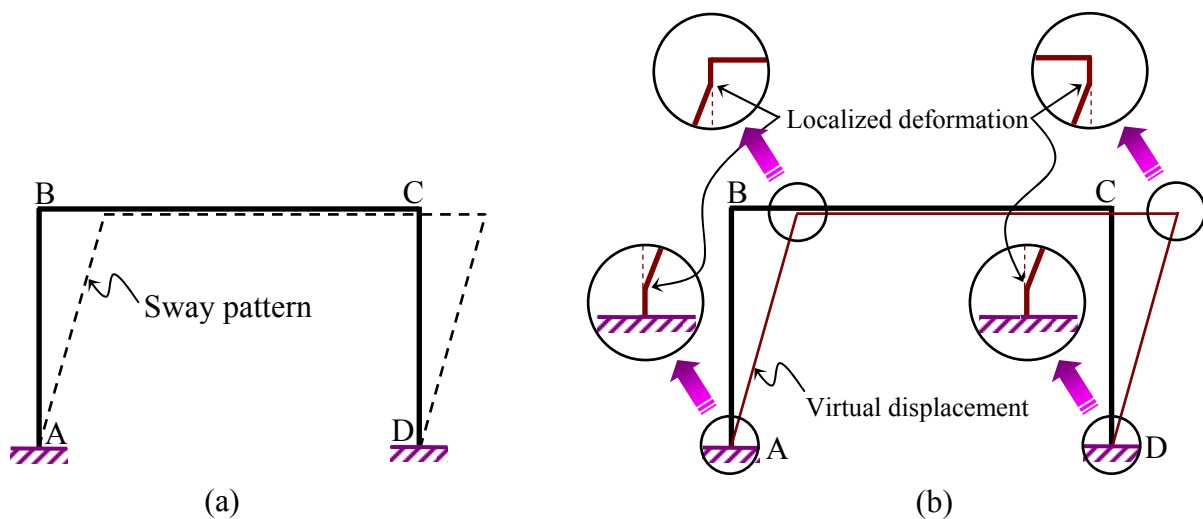


Figure 12.21: Schematic of (a) sway pattern and (b) virtual displacement

The external virtual work due to all applied loads, denoted by  $\delta W_{\text{ext}}$ , can be computed by using following three results: (i) the external virtual work due to a concentrated force is equal to the product of that force and the *corresponding* virtual displacement, (ii) the external virtual work due to a distributed force is equal to the product of its resultant and the *corresponding* virtual displacement at the location of its resultant, and (iii) the external virtual work due to a concentrated moment within the member is equal to the product of that moment and the *corresponding* virtual rotation. The adjective *corresponding* is used to emphasize that the displacement or rotation is considered in the direction and at the location of the applied load or its resultant. It is also important to emphasize that concentrated moments applied directly to nodes do not contribute to the external virtual work since rotations at all nodes of the chosen virtual displacement vanish. Once the external virtual work and the internal virtual work are obtained, the equilibrium equation for the  $i^{\text{th}}$  sway degree of freedom is given by

$$\delta W_{\text{int}} = \delta W_{\text{ext}} \quad (12.72)$$

To clearly demonstrate above procedure, let us consider again the frame shown in Figure 12.16(a). The sway pattern obtained from the ICR concept (and used as a virtual displacement) is shown again in Figure 12.22 along with applied loads. The sway angles for members AB, BC and CD in terms of the sway degree of freedom  $\Delta_1$  are summarized again here as  $\psi_{AB} = -\Delta_1/L$ ,  $\psi_{BC} = 2\Delta_1/3L$  and  $\psi_{CD} = \Delta_1/L$ . By using equation (12.71), the internal virtual work  $\delta W_{\text{int}}$  for this particular frame becomes

$$\begin{aligned} \delta W_{\text{int}} &= -(M_{AB} + M_{BA})\psi_{AB} - (M_{BC} + M_{CB})\psi_{BC} - (M_{CD} + M_{DC})\psi_{CD} \\ &= (M_{AB} + M_{BA})\Delta_1 / L - (M_{BC} + M_{CB})(2\Delta_1 / 3L) + (M_{CD} + M_{DC})\Delta_1 / L \end{aligned}$$

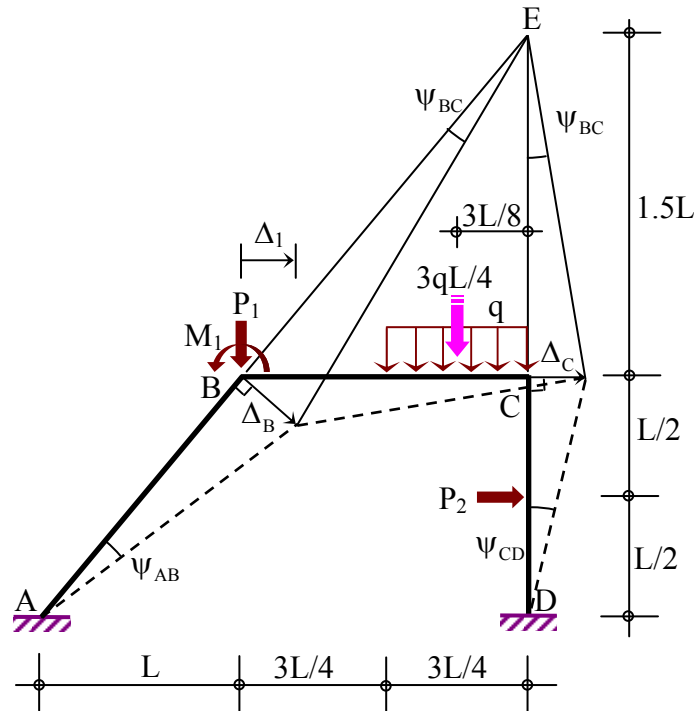


Figure 12.22: Schematic of the virtual displacement obtained from the sway pattern

The external virtual work  $\delta W_{\text{ext}}$  due to the concentrated forces  $P_1$  and  $P_2$  and the distributed force  $q$  is given by

$$\delta W_{\text{ext}} = P_1(\psi_{AB})(L) + P_2(\psi_{CD})(L/2) + (3qL/4)(\psi_{BC})(3L/8) = P_1\Delta_1 + P_2\Delta_1/2 + 3qL\Delta_1/16$$

Note again that a moment  $M_1$  applied to node B does not contribute to  $\delta W_{\text{ext}}$ . An equilibrium equation associated with the sway degree of freedom is then obtained by equating  $\delta W_{\text{ext}}$  and  $\delta W_{\text{int}}$  and then cancelling  $\Delta_1$ :

$$\delta W_{\text{int}} = \delta W_{\text{ext}} \Rightarrow (M_{AB} + M_{BA})/L - 2(M_{BC} + M_{CB})/3L + (M_{CD} + M_{DC})/L = P_1 + P_2/2 + 3qL/16$$

For this particular frame which contains three primary unknowns  $\{\theta_B, \theta_C, \Delta_1\}$ , three independent equilibrium equations sufficient for solving all these unknowns are summarized again below:

$$M_{BA} + M_{BC} - M_1 = 0$$

$$M_{CB} + M_{CD} = 0$$

$$(M_{AB} + M_{BA})/L - 2(M_{BC} + M_{CB})/3L + (M_{CD} + M_{DC})/L = P_1 + P_2/2 + 3qL/16$$

where the first two equations are associated with moment equilibrium at nodes B and C and the last equation is an equilibrium equation corresponding to the sway degree of freedom. These three equations can be further expressed in terms of the primary unknowns by employing the slope-deflection equations obtained previously for each member.

### 12.6.6 Solve for primary unknowns and determine other quantities

After a complete set of equilibrium equations is established, all primary unknowns can be obtained by solving a system of linear algebraic equations. Once all primary unknowns are determined, other quantities such as the end moments, the end shear forces, the end axial forces, and support reactions can be computed. For instance, the end moments of each member can be obtained from the slope-deflection equations (12.15) and (12.16) by substituting the solved end rotations and sway angle. Once the end moments are known, the end shear forces can be computed from two equilibrium equations such as  $\Sigma M_{\text{end A}} = 0$  and  $\Sigma M_{\text{end B}} = 0$  or  $\Sigma M_{\text{end A}} = 0$  and  $\Sigma F_y = 0$  (see the free body diagram of a generic member AB in Figure 12.23 for clarification).

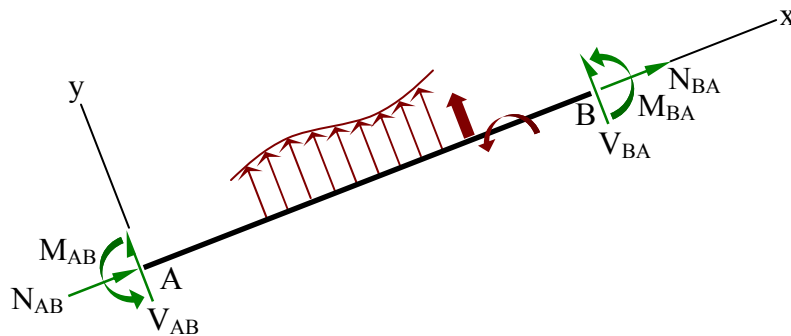


Figure 12.23: Free body diagram of generic member AB

The axial force at both ends of the member can not be obtained directly from force equilibrium of such member but it is required to consider force equilibrium at nodes (see detail procedure in following examples). After end forces and end moments are determined, all reactions can be obtained by considering equilibrium at supports. Finally, the axial force diagram (AFD), the shear force diagram (SFD) and the bending moment diagram (BMD) for all member can be constructed using standard procedure as discussed in Chapter 2. The displacement and rotation at any point within the member can also be computed using various techniques such as the method of moment area, the method of conjugate structure analogy, the unit load method, or even the slope-deflection equations.

All essential steps in the analysis of structures by the slope-deflection equations as described above are summarized again in a form of a diagram shown in Figure 12.24 in order to see the overall picture of the method.

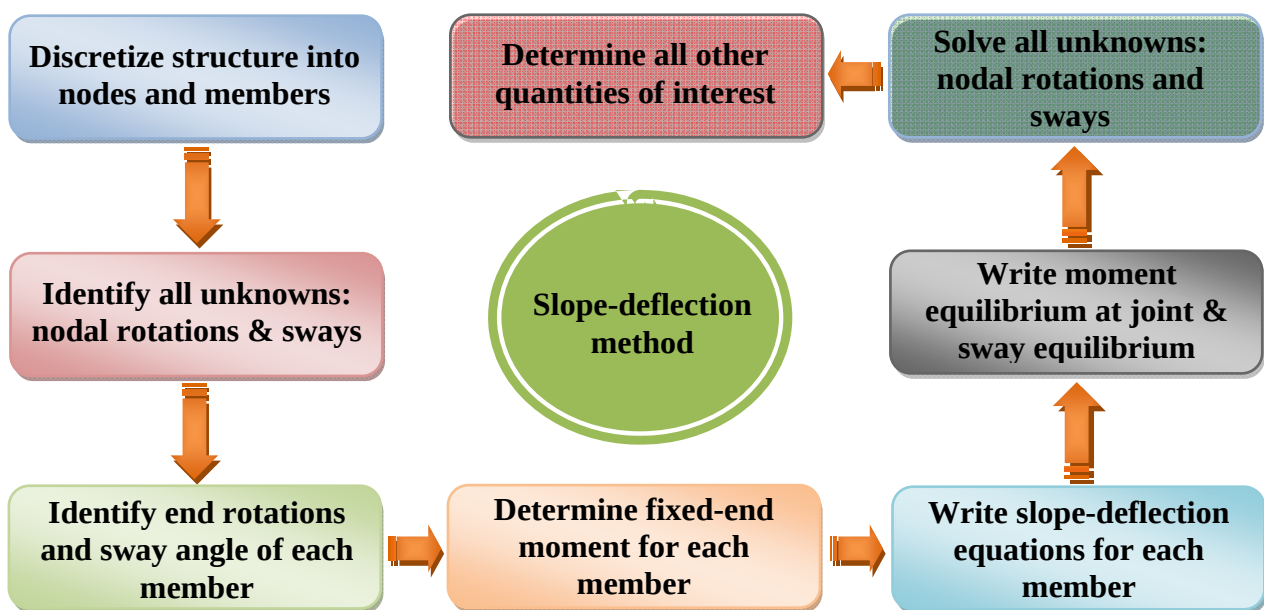
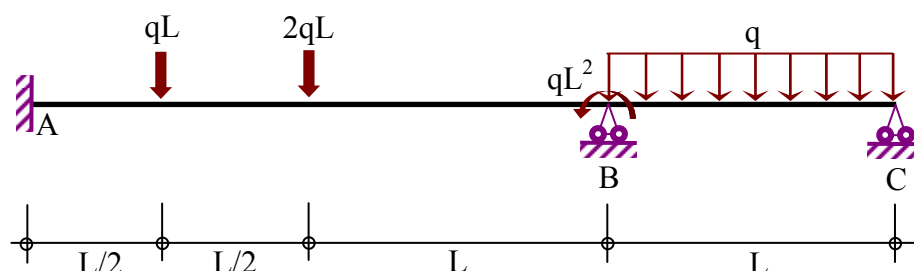


Figure 12.24 Diagram showing all essential steps in analysis of structures by slope-deflection equations

**Example 12.9** Use a method of slope-deflection equations to analyze a continuous beam under external loads shown below for two cases: (1) no support settlement and (2) supports B and C are subjected to downward settlements  $\Delta_0$  and  $2\Delta_0$ , respectively. The Young modulus  $E$  and the moment of inertia  $I$  are assumed to be constant throughout the structure.





**Solution** Due to the geometry and flexural rigidity of the given beam, it is discretized into two members (i.e. members AB and BC) with three nodes (i.e. nodes A, B, and C). The number of rotational degrees of freedom is given by  $N_r = 3 + 0 - 1 = 2$  while the number of sway degrees of freedom is given by  $N_s = 3 - 3 = 0$ . Therefore, the number of primary unknowns is equal to 2 and they are the rotations at nodes B and C (i.e.  $\theta_B$  and  $\theta_C$ ).

Now, let us consider the first case where there is no support settlement. The end rotations, the sway angle, the fixed-end moments, and the two slope-deflection equations for the members AB and BC are given below.

Member AB For this member,  $\theta_A = 0$ ,  $\psi_{AB} = 0$  and

$$FEM_{AB} = \frac{qL(L/2)(3L/2)^2}{(2L)^2} + \frac{2qL(2L)}{8} = \frac{25qL^2}{32}$$

$$FEM_{BA} = -\frac{qL(3L/2)(L/2)^2}{(2L)^2} - \frac{2qL(2L)}{8} = -\frac{19qL^2}{32}$$

The slope-deflection equations become

$$M_{AB} = \frac{2EI}{2L}(2 \cdot 0 + \theta_B) - \frac{6EI(0)}{2L} + \frac{25qL^2}{32} = \frac{EI\theta_B}{L} + \frac{25qL^2}{32} \quad (e12.9.1)$$

$$M_{BA} = \frac{2EI}{2L}(0 + 2\theta_B) - \frac{6EI(0)}{2L} - \frac{19qL^2}{32} = \frac{2EI\theta_B}{L} - \frac{19qL^2}{32} \quad (e12.9.2)$$

Member BC For this member,  $\psi_{BC} = 0$  and

$$FEM_{BC} = \frac{qL^2}{12}$$

$$FEM_{CB} = -\frac{qL^2}{12}$$

The slope-deflection equations become

$$M_{BC} = \frac{2EI}{L}(2\theta_B + \theta_C) - \frac{6EI(0)}{L} + \frac{qL^2}{12} = \frac{4EI\theta_B}{L} + \frac{2EI\theta_C}{L} + \frac{qL^2}{12} \quad (e12.9.3)$$

$$M_{CB} = \frac{2EI}{L}(2\theta_C + \theta_B) - \frac{6EI(0)}{L} - \frac{qL^2}{12} = \frac{2EI\theta_B}{L} + \frac{4EI\theta_C}{L} - \frac{qL^2}{12} \quad (e12.9.4)$$

To solve for the two primary unknowns  $\theta_B$  and  $\theta_C$ , it is required to set up two equilibrium equations and for this particular case (the discretized structure is non-sway), these two equations come directly from moment equilibrium of nodes B and C. The slope-deflection equations obtained above are used to express such equilibrium equations in terms of the primary unknowns  $\theta_B$  and  $\theta_C$  as demonstrated below.

Moment equilibrium at node B

$$\begin{aligned} [\Sigma M_{\text{node B}} = 0] &\Rightarrow M_{BA} + M_{BC} - qL^2 = 0 \\ &\Rightarrow \left( \frac{2EI\theta_B}{L} - \frac{19qL^2}{32} \right) + \left( \frac{4EI\theta_B}{L} + \frac{2EI\theta_C}{L} + \frac{qL^2}{12} \right) = qL^2 \\ &\Rightarrow \frac{6EI\theta_B}{L} + \frac{2EI\theta_C}{L} = \frac{145qL^2}{96} \end{aligned} \quad (e12.9.5)$$



### Moment equilibrium at node C

$$\begin{aligned} [\Sigma M_{\text{node C}} = 0] &\Rightarrow M_{CB} = 0 \\ &\Rightarrow \frac{2EI\theta_B}{L} + \frac{4EI\theta_C}{L} = \frac{qL^2}{12} \end{aligned} \quad (\text{e12.9.6})$$

By solving two linear equations (e12.9.5) and (e12.9.6) simultaneously, it leads to

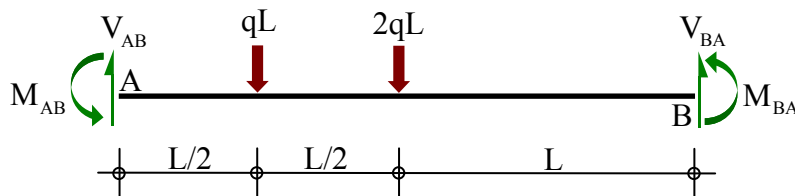
$$\frac{EI\theta_B}{L} = 0.2938qL^2 \quad (\text{e12.9.7})$$

$$\frac{EI\theta_C}{L} = -0.1260qL^2 \quad (\text{e12.9.8})$$

Once the primary unknowns ( $\theta_B$  and  $\theta_C$ ) are solved, the end moments of each member can readily be obtained by inserting  $\theta_B$  and  $\theta_C$  into the slope-deflection equations (e12.9.1)-(e12.9.4) and, subsequently, the end shear forces can directly be computed from equilibrium of the member as shown below.

### Member AB

Free body diagram:



End moments:

$$(\text{e12.9.1}) \Rightarrow M_{AB} = \frac{EI\theta_B}{L} + \frac{25qL^2}{32} = 0.2938qL^2 + \frac{25qL^2}{32} = 1.075qL^2$$

$$(\text{e12.9.2}) \Rightarrow M_{BA} = \frac{2EI\theta_B}{L} - \frac{19qL^2}{32} = 2(0.2938qL^2) - \frac{19qL^2}{32} = -0.00625qL^2$$

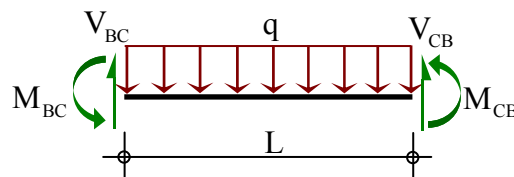
End shear forces:

$$[\Sigma M_{\text{end B}} = 0] \Rightarrow V_{AB}(2L) - M_{AB} - M_{BA} - qL(3L/2) - 2qL(L) = 0 \Rightarrow V_{AB} = 2.2844qL$$

$$[\Sigma F_{Y, \text{member AB}} = 0] \Rightarrow V_{AB} + V_{BA} - qL - 2qL = 0 \Rightarrow V_{BA} = 0.7156qL$$

### Member BC

Free body diagram:



End moments:

$$(\text{e12.9.3}) \Rightarrow M_{BC} = \frac{4EI\theta_B}{L} + \frac{2EI\theta_C}{L} + \frac{qL^2}{12} = 4(0.2938qL^2) + 2(-0.1260qL^2) + \frac{qL^2}{12} = 1.00625qL^2$$

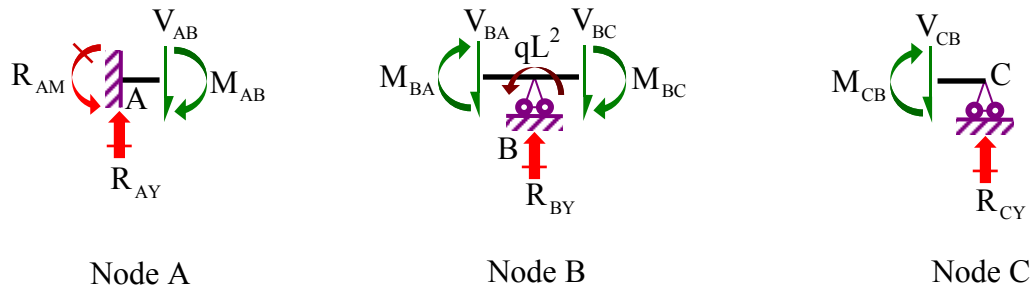
$$(e12.9.4) \Rightarrow M_{CB} = \frac{2EI\theta_B}{L} + \frac{4EI\theta_C}{L} - \frac{qL^2}{12} = 2(0.2938qL^2) + 4(-0.1260qL^2) - \frac{qL^2}{12} = 0$$

End shear forces:

$$[\Sigma M_{\text{end C}} = 0] \Rightarrow V_{BC}(L) - M_{BC} - M_{CB} - qL(L/2) = 0 \Rightarrow V_{BC} = 1.50625qL$$

$$[\Sigma F_{Y, \text{member BC}} = 0] \Rightarrow V_{BC} + V_{CB} - qL = 0 \Rightarrow V_{CB} = -0.50625qL$$

All support reactions are obtained by enforcing equilibrium of nodes A, B and C as indicated below.

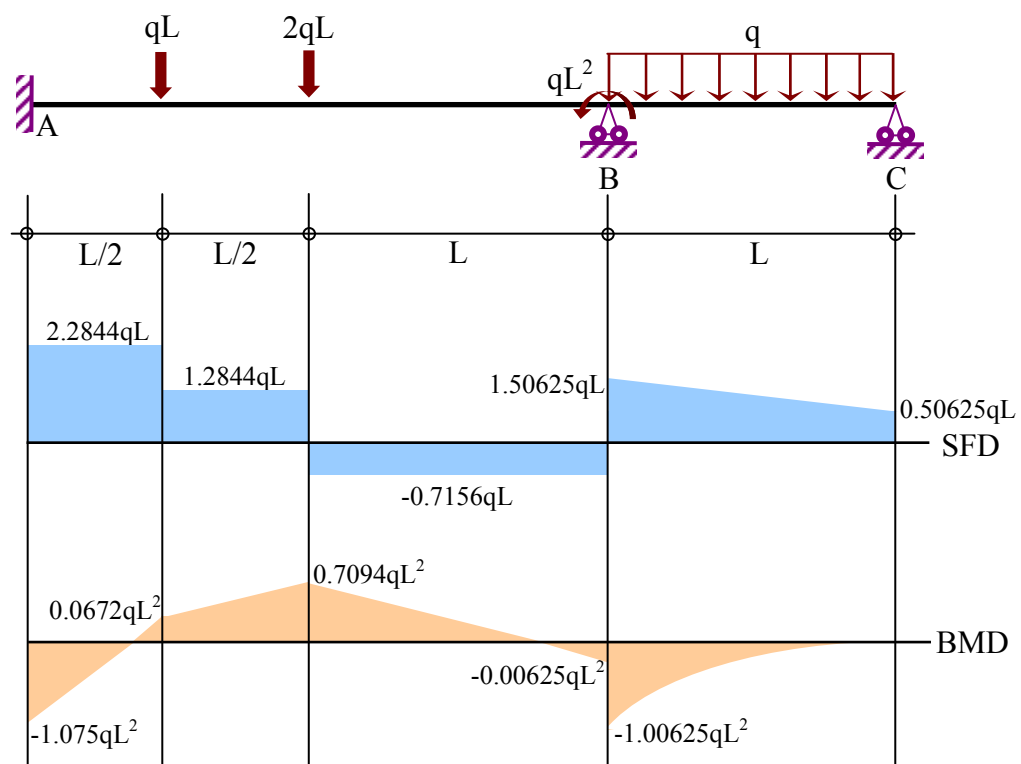


Node A:  $R_{AY} = V_{AB} = 2.2844qL$  Upward ;  $R_{AM} = M_{AB} = 1.075qL^2$  CCW

Node B:  $R_{BY} = V_{BA} + V_{BC} = 2.2219qL$  Upward

Node C:  $R_{CY} = V_{CB} = -0.5063qL$  Downward

The shear force diagram (SFD) and bending moment diagram (BMD) of the beam are shown below



Next, let us consider the second case where points B and C are subjected to *prescribed* downward settlements. It should be remarked that while support settlements at points B and C induce non-zero sway angle to members AB and BC, there is still no sway degree of freedom. This is due to that nodes B and C are not allowed to move freely. The slope-deflection equations for the two members for this particular case are slightly different from the first case in that the known, non-zero sway angles must be included as shown below.

Member AB For this member,  $\theta_A = 0$ ,  $\psi_{AB} = -\Delta_0/2L$ ,  $FEM_{AB} = 25qL^2/32$ , and  $FEM_{BA} = -19qL^2/32$ . The slope-deflection equations now become

$$M_{AB} = \frac{EI\theta_B}{L} - \frac{6EI}{2L} \left( -\frac{\Delta_0}{2L} \right) + \frac{25qL^2}{32} = \frac{EI\theta_B}{L} + \frac{3EI\Delta_0}{2L^2} + \frac{25qL^2}{32} \quad (e12.9.9)$$

$$M_{BA} = \frac{2EI\theta_B}{L} - \frac{6EI}{2L} \left( -\frac{\Delta_0}{2L} \right) - \frac{19qL^2}{32} = \frac{2EI\theta_B}{L} + \frac{3EI\Delta_0}{2L^2} - \frac{19qL^2}{32} \quad (e12.9.10)$$

Member BC For this member,  $\psi_{BC} = -(2\Delta_0 - \Delta_0)/L = -\Delta_0/L$ ,  $FEM_{BC} = qL^2/12$ , and  $FEM_{CB} = -qL^2/12$ . The slope-deflection equations now become

$$M_{BC} = \frac{4EI\theta_B}{L} + \frac{2EI\theta_C}{L} - \frac{6EI}{L} \left( -\frac{\Delta_0}{L} \right) + \frac{qL^2}{12} = \frac{4EI\theta_B}{L} + \frac{2EI\theta_C}{L} + \frac{6EI\Delta_0}{L^2} + \frac{qL^2}{12} \quad (e12.9.11)$$

$$M_{CB} = \frac{2EI\theta_B}{L} + \frac{4EI\theta_C}{L} - \frac{6EI}{L} \left( -\frac{\Delta_0}{L} \right) - \frac{qL^2}{12} = \frac{2EI\theta_B}{L} + \frac{4EI\theta_C}{L} + \frac{6EI\Delta_0}{L^2} - \frac{qL^2}{12} \quad (e12.9.12)$$

The two moment equilibrium equations at nodes B and C in terms of the primary unknowns  $\theta_B$  and  $\theta_C$  become

Moment equilibrium at node B

$$\begin{aligned} [\Sigma M_{\text{node B}} = 0] &\Rightarrow M_{BA} + M_{BC} - qL^2 = 0 \\ &\Rightarrow \left( \frac{2EI\theta_B}{L} + \frac{3EI\Delta_0}{2L^2} - \frac{19qL^2}{32} \right) + \left( \frac{4EI\theta_B}{L} + \frac{2EI\theta_C}{L} + \frac{6EI\Delta_0}{L^2} + \frac{qL^2}{12} \right) = qL^2 \\ &\Rightarrow \frac{6EI\theta_B}{L} + \frac{2EI\theta_C}{L} = \frac{145qL^2}{96} - \frac{15EI\Delta_0}{2L^2} \end{aligned} \quad (e12.9.13)$$

Moment equilibrium at node C

$$\begin{aligned} [\Sigma M_{\text{node C}} = 0] &\Rightarrow M_{CB} = 0 \\ &\Rightarrow \frac{2EI\theta_B}{L} + \frac{4EI\theta_C}{L} = \frac{qL^2}{12} - \frac{6EI\Delta_0}{L^2} \end{aligned} \quad (e12.9.14)$$

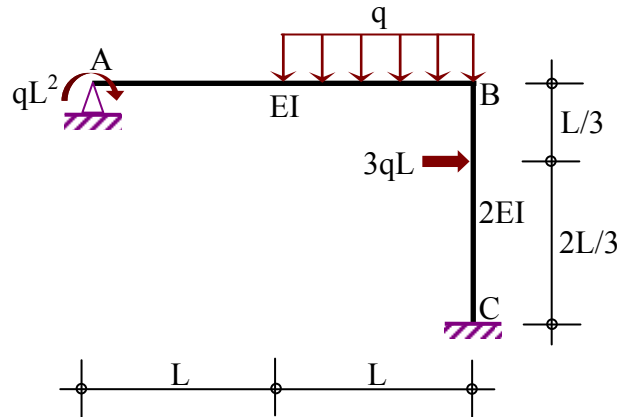
By solving equations (e12.9.13) and (e12.9.14) simultaneously, we obtain

$$\frac{EI\theta_B}{L} = 0.2938qL^2 - \frac{0.9EI\Delta_0}{L^2} \quad (e12.9.15)$$

$$\frac{EI\theta_C}{L} = -0.1260qL^2 - \frac{1.05EI\Delta_0}{L^2} \quad (e12.9.16)$$

After  $\theta_B$  and  $\theta_C$  are solved, the end moments and end shear forces of members AB and BC, support reactions, and SFD and BMD can readily be obtained in the same manner as the first case.

**Example 12.10** Use a method of slope-deflection equations to analyze a rigid frame under external loads shown below. The flexural rigidities of all members are indicated in the figure.



**Solution** To minimize the number of primary unknowns, the given frame is discretized into two members (i.e. members AB and BC) with three nodes (e.g. nodes A, B, C). The number of rotational degrees of freedom is given by  $N_r = 3 + 0 - 1 = 2$  while the number of sway degrees of freedom is given by  $N_s = 2(3) - 4 - 2 = 0$ . Therefore, the number of primary unknowns is equal to 2 and they are the rotations at nodes A and B (i.e.  $\theta_A$  and  $\theta_B$ ).

The end rotations, the sway angle, the fixed-end moments, and the slope-deflection equations for the members AB and BC are given below.

**Member AB** For this member,  $\psi_{AB} = 0$  and

$$FEM_{AB} = \frac{q(2L)^2}{12} \left( 4(1/2)^3 - 3(1/2)^4 \right) = \frac{5qL^2}{48}$$

$$FEM_{BA} = -\frac{q(2L)^2}{12} \left( 6(1/2)^2 - 8(1/2)^3 + 3(1/2)^4 \right) = -\frac{11qL^2}{48}$$

The slope-deflection equations become

$$M_{AB} = \frac{2EI}{2L} (2\theta_A + \theta_B) - \frac{6EI(0)}{2L} + \frac{5qL^2}{48} = \frac{2EI\theta_A}{L} + \frac{EI\theta_B}{L} + \frac{5qL^2}{48} \quad (e12.10.1)$$

$$M_{BA} = \frac{2EI}{2L} (\theta_A + 2\theta_B) - \frac{6EI(0)}{2L} - \frac{11qL^2}{48} = \frac{EI\theta_A}{L} + \frac{2EI\theta_B}{L} - \frac{11qL^2}{48} \quad (e12.10.2)$$

**Member BC** For this member,  $\theta_C = 0$ ,  $\psi_{BC} = 0$  and

$$FEM_{BC} = -\frac{(3qL)(L/3)(2L/3)^2}{L^2} = -\frac{4qL^2}{9}$$

$$FEM_{CB} = \frac{(3qL)(2L/3)(L/3)^2}{L^2} = \frac{2qL^2}{9}$$

The slope-deflection equations become

$$M_{BC} = \frac{2(2EI)}{L} (2\theta_B + 0) - \frac{6(2EI)(0)}{L} - \frac{4qL^2}{9} = \frac{8EI\theta_B}{L} - \frac{4qL^2}{9} \quad (e12.10.3)$$

$$M_{CB} = \frac{2(2EI)}{L} (\theta_B + 0) - \frac{6(2EI)(0)}{L} + \frac{2qL^2}{9} = \frac{4EI\theta_B}{L} + \frac{2qL^2}{9} \quad (e12.10.4)$$

Since the discretized structure is non-sway, two equilibrium equations in terms of  $\theta_A$  and  $\theta_B$  can be obtained directly by forming moment equilibrium of nodes A and B and then applying above slope-deflection equations as follows.

Moment equilibrium at node A

$$\begin{aligned} [\Sigma M_{\text{node A}} = 0] &\Rightarrow M_{AB} + qL^2 = 0 \\ &\Rightarrow \frac{2EI\theta_A}{L} + \frac{EI\theta_B}{L} + \frac{5qL^2}{48} + qL^2 = 0 \\ &\Rightarrow \frac{2EI\theta_A}{L} + \frac{EI\theta_B}{L} = -\frac{53qL^2}{48} \end{aligned} \quad (\text{e12.10.5})$$

Moment equilibrium at node B

$$\begin{aligned} [\Sigma M_{\text{node B}} = 0] &\Rightarrow M_{BA} + M_{BC} = 0 \\ &\Rightarrow \left( \frac{EI\theta_A}{L} + \frac{2EI\theta_B}{L} - \frac{11qL^2}{48} \right) + \left( \frac{8EI\theta_B}{L} - \frac{4qL^2}{9} \right) = 0 \\ &\Rightarrow \frac{EI\theta_A}{L} + \frac{10EI\theta_B}{L} = \frac{97qL^2}{144} \end{aligned} \quad (\text{e12.10.6})$$

By solving equations (e12.10.5) and (e12.10.6) simultaneously, we obtain

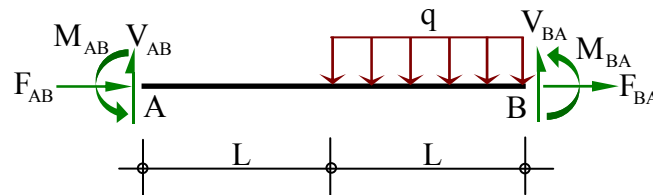
$$\frac{EI\theta_A}{L} = -0.6166qL^2 \quad (\text{e12.10.7})$$

$$\frac{EI\theta_B}{L} = 0.1290qL^2 \quad (\text{e12.10.8})$$

After the nodal rotations  $\theta_B$  and  $\theta_C$  are determined, the end moments and end shear forces of members AB and BC can be computed as follows.

Member AB

Free body diagram:



End moments:

$$(\text{e12.10.1}) \Rightarrow M_{AB} = \frac{2EI\theta_A}{L} + \frac{EI\theta_B}{L} + \frac{5qL^2}{48} = 2(-0.6166qL^2) + 0.1290qL^2 + \frac{5qL^2}{48} = -qL^2$$

$$(\text{e12.10.2}) \Rightarrow M_{BA} = \frac{EI\theta_A}{L} + \frac{2EI\theta_B}{L} - \frac{11qL^2}{48} = -0.6166qL^2 + 2(0.1290qL^2) - \frac{11qL^2}{48} = -0.588qL^2$$

End shear forces:

$$[\Sigma M_{\text{end B}} = 0] \Rightarrow V_{AB}(2L) - M_{AB} - M_{BA} - qL(L/2) = 0 \Rightarrow V_{AB} = -0.544qL$$

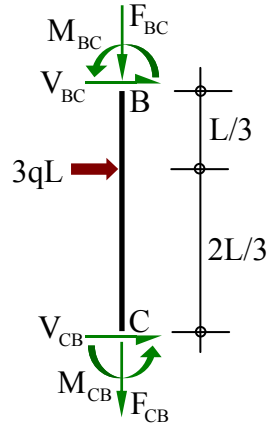
$$\left[ \Sigma F_{y, \text{ member AB}} = 0 \right] \Rightarrow V_{AB} + V_{BA} - qL = 0 \Rightarrow V_{BA} = 1.544qL$$

End axial forces:

$$\left[ \Sigma F_{x, \text{ member AB}} = 0 \right] \Rightarrow F_{AB} + F_{BA} = 0 \quad (\text{e12.10.9})$$

### Member BC

Free body diagram:



End moments:

$$(\text{e12.10.3}) \Rightarrow M_{BC} = \frac{8EI\theta_B}{L} - \frac{4qL^2}{9} = 8(0.1290qL^2) - \frac{4qL^2}{9} = 0.588qL^2$$

$$(\text{e12.10.4}) \Rightarrow M_{CB} = \frac{4EI\theta_B}{L} + \frac{2qL^2}{9} = 4(0.1290qL^2) + \frac{2qL^2}{9} = 0.738qL^2$$

End shear forces:

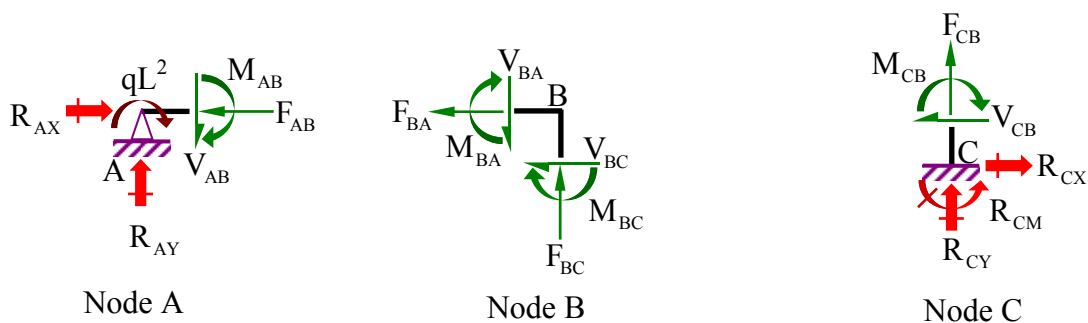
$$\left[ \Sigma M_{\text{end C}} = 0 \right] \Rightarrow V_{BC}(L) - M_{BC} - M_{CB} + 3qL(2L/3) = 0 \Rightarrow V_{BC} = -0.674qL$$

$$\left[ \Sigma F_{y, \text{ member BC}} = 0 \right] \Rightarrow V_{BC} + V_{CB} + 3qL = 0 \Rightarrow V_{CB} = -2.326qL$$

End axial forces:

$$\left[ \Sigma F_{x, \text{ member BC}} = 0 \right] \Rightarrow F_{BC} + F_{CB} = 0 \quad (\text{e12.10.10})$$

It should be noted that enforcing equilibrium of members does not yield the end axial forces but only provides their relation. Now, by considering equilibrium of all nodes, we can obtain sufficient equations to determine the end axial forces and also all support reactions.



Node C:  $F_{BC} = V_{BA} = 1.544qL$  ;  $F_{BA} = -V_{BC} = 0.674qL$

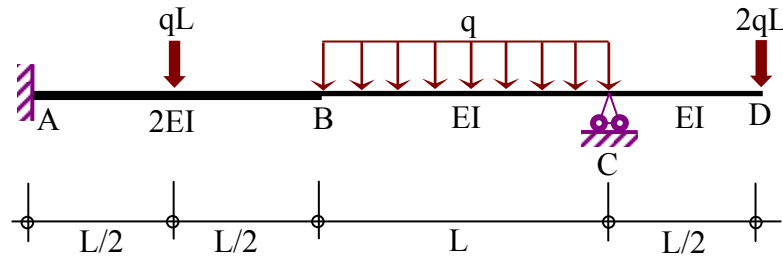
$$(e12.10.9) \Rightarrow F_{AB} = -F_{BA} = -0.674qL \quad \text{and} \quad (e12.10.10) \Rightarrow F_{CB} = -F_{BC} = -1.544qL$$

Node A:  $R_{AX} = F_{AB} = -0.674qL$  Leftward ;  $R_{AY} = V_{AB} = -0.544qL$  Downward

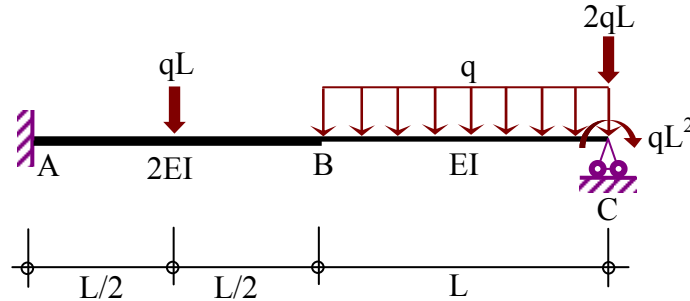
Node E:  $R_{CX} = V_{CB} = -2.326qL$  Leftward ;  $R_{CY} = -F_{CB} = 1.544qL$  Upward ;  
 $R_{CM} = M_{CB} = 0.738qL^2$  CCW

The AFD, SFD and BMD for members AB and BC can also be obtained using standard procedure.

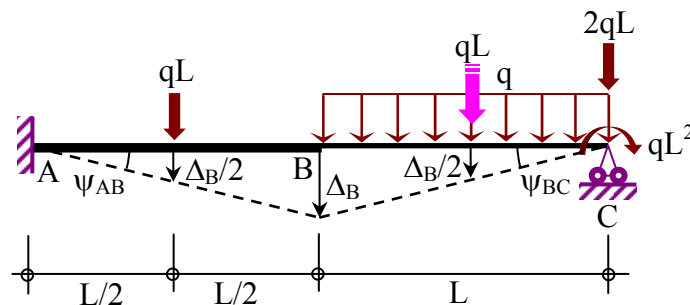
**Example 12.11** Use a method of slope-deflection equations to analyze a beam under external loads shown below. The flexural rigidity of segments AB, BC and CD are given by  $2EI$ ,  $EI$  and  $EI$ , respectively.



**Solution** Since the segment CD is a statically determinate segment (i.e. the shear force and bending moment at every point within this segment can readily be obtained from static equilibrium), a reduced structure equivalent to the given beam can be obtained by eliminating the segment CD and then replacing it by a set of force and moment acting at point C as shown below.



Now, let us discretize the beam shown above using the minimum member strategy. Since the segment ABC is not prismatic, a point B must be treated as a node. The discretized structure therefore consists of three nodes (i.e. nodes A, B, and C) and two members (i.e. members AB and BC). The number of rotational degrees of freedom is given by  $N_r = 3 + 0 - 1 = 2$  (i.e.  $\theta_B$  and  $\theta_C$ ) whereas the number of sway degrees of freedom is equal to  $N_s = 3 - 2 = 1$  (the deflection at point B, denoted by  $\Delta_B$ , is chosen to represent this sway degree of freedom). Therefore, the total number of primary unknowns is equal to  $2 + 1 = 3$  (i.e.  $\theta_B$ ,  $\theta_C$  and  $\Delta_B$ ). The sway pattern associated with the sway degree of freedom is sketched as shown below.



The end rotations, the sway angle in terms of the sway degree of freedom  $\Delta_B$ , the fixed-end moments, and the slope-deflection equations for the members AB and BC are given below.

Member AB For this member,  $\theta_A = 0$ ,  $\psi_{AB} = -\Delta_B/L$  and

$$FEM_{AB} = \frac{(qL)L}{8} = \frac{qL^2}{8} \quad ; \quad FEM_{BA} = -\frac{(qL)L}{8} = -\frac{qL^2}{8}$$

The slope-deflection equations become

$$M_{AB} = \frac{2(2EI)}{L}(2 \cdot 0 + \theta_B) - \frac{6(2EI)}{L}\left(-\frac{\Delta_B}{L}\right) + \frac{qL^2}{8} = \frac{4EI\theta_B}{L} + \frac{12EI\Delta_B}{L^2} + \frac{qL^2}{8} \quad (e12.11.1)$$

$$M_{BA} = \frac{2(2EI)}{L}(0 + 2\theta_B) - \frac{6(2EI)}{L}\left(-\frac{\Delta_B}{L}\right) - \frac{qL^2}{8} = \frac{8EI\theta_B}{L} + \frac{12EI\Delta_B}{L^2} - \frac{qL^2}{8} \quad (e12.11.2)$$

Member BC For this member,  $\psi_{BC} = \Delta_B/L$  and

$$FEM_{BC} = \frac{qL^2}{12} \quad ; \quad FEM_{CB} = -\frac{qL^2}{12}$$

The slope-deflection equations become

$$M_{BC} = \frac{2EI}{L}(2\theta_B + \theta_C) - \frac{6EI}{L}\left(\frac{\Delta_B}{L}\right) + \frac{qL^2}{12} = \frac{4EI\theta_B}{L} + \frac{2EI\theta_C}{L} - \frac{6EI\Delta_B}{L^2} + \frac{qL^2}{12} \quad (e12.11.3)$$

$$M_{CB} = \frac{2EI}{L}(\theta_B + 2\theta_C) - \frac{6EI}{L}\left(\frac{\Delta_B}{L}\right) - \frac{qL^2}{12} = \frac{2EI\theta_B}{L} + \frac{4EI\theta_C}{L} - \frac{6EI\Delta_B}{L^2} - \frac{qL^2}{12} \quad (e12.11.4)$$

Next, we need to set up three equilibrium equations to solve for three primary unknowns  $\theta_B$ ,  $\theta_C$  and  $\Delta_B$ ; two of them are obtained from moment equilibrium at nodes B and C and the third one is an equilibrium equation associated with the sway mode. The first two equations are obtained as follows.

Moment equilibrium at node B

$$\begin{aligned} [\Sigma M_{\text{node B}} = 0] &\Rightarrow M_{BA} + M_{BC} = 0 \\ &\Rightarrow \left(\frac{8EI\theta_B}{L} + \frac{12EI\Delta_B}{L^2} - \frac{qL^2}{8}\right) + \left(\frac{4EI\theta_B}{L} + \frac{2EI\theta_C}{L} - \frac{6EI\Delta_B}{L^2} + \frac{qL^2}{12}\right) = 0 \\ &\Rightarrow \frac{12EI\theta_B}{L} + \frac{2EI\theta_C}{L} + \frac{6EI\Delta_B}{L^2} = \frac{qL^2}{24} \end{aligned} \quad (e12.11.5)$$

Moment equilibrium at node C

$$\begin{aligned} [\Sigma M_{\text{node C}} = 0] &\Rightarrow M_{CB} + qL^2 = 0 \\ &\Rightarrow \frac{2EI\theta_B}{L} + \frac{4EI\theta_C}{L} - \frac{6EI\Delta_B}{L^2} - \frac{qL^2}{12} + qL^2 = 0 \\ &\Rightarrow \frac{2EI\theta_B}{L} + \frac{4EI\theta_C}{L} - \frac{6EI\Delta_B}{L^2} = -\frac{11qL^2}{12} \end{aligned} \quad (e12.11.6)$$

The last equation is obtained by applying the principle of virtual work with the displacement associated with the sway pattern being chosen as the virtual displacement. The calculation of the



external virtual work and the internal virtual work and the setup of a sway equilibrium equation are shown below.

$$\delta W_{\text{ext}} = (qL)(\Delta_B / 2) + (qL)(\Delta_B / 2) = qL\Delta_B$$

$$\delta W_{\text{int}} = -(M_{AB} + M_{BA})\psi_{AB} - (M_{BC} + M_{CB})\psi_{BC} = (M_{AB} + M_{BA})\Delta_B / L - (M_{BC} + M_{CB})\Delta_B / L$$

$$\delta W_{\text{ext}} = \delta W_{\text{int}} \Rightarrow qL\Delta_B = (M_{AB} + M_{BA})\Delta_B / L - (M_{BC} + M_{CB})\Delta_B / L$$

$$\Rightarrow M_{AB} + M_{BA} - M_{BC} - M_{CB} = qL^2$$

$$\Rightarrow \frac{6EI\theta_B}{L} - \frac{6EI\theta_C}{L} + \frac{36EI\Delta_B}{L^2} = qL^2 \quad (\text{e12.11.7})$$

By solving a system of three linear equations (e12.11.5), (e12.11.6) and (e12.11.7), we obtain

$$\frac{EI\theta_B}{L} = 0.0781qL^2 \quad (\text{e12.11.8})$$

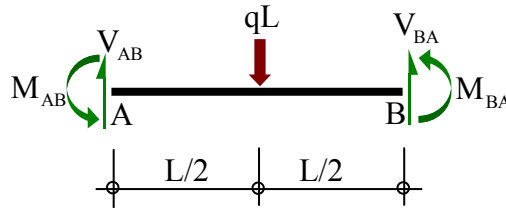
$$\frac{EI\theta_C}{L} = -0.3281qL^2 \quad (\text{e12.11.9})$$

$$\frac{EI\Delta_B}{L^2} = -0.0399qL^2 \quad (\text{e12.11.10})$$

Once all primary unknowns are solved, the end moments of members AB and BC are obtained from the slope-deflection equations (e12.11.1)-(e12.11.4) and the end shear forces can be computed from equilibrium of members as shown below.

#### Member AB

Free body diagram:



End moments:

$$(\text{e12.11.1}) \Rightarrow M_{AB} = \frac{4EI\theta_B}{L} + \frac{12EI\Delta_B}{L^2} + \frac{qL^2}{8} = 4(0.0781qL^2) + 12(-0.0399qL^2) + \frac{qL^2}{8} = -0.0417qL^2$$

$$(\text{e12.11.2}) \Rightarrow M_{BA} = \frac{8EI\theta_B}{L} + \frac{12EI\Delta_B}{L^2} - \frac{qL^2}{8} = 8(0.0781qL^2) + 12(-0.0399qL^2) - \frac{qL^2}{8} = 0.0208qL^2$$

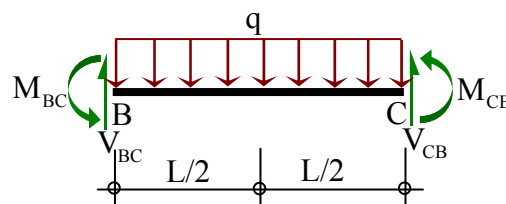
End shear forces:

$$[\Sigma M_{\text{end B}} = 0] \Rightarrow V_{AB}(L) - M_{AB} - M_{BA} - qL(L/2) = 0 \Rightarrow V_{AB} = 0.479qL$$

$$[\Sigma F_{y, \text{ member AB}} = 0] \Rightarrow V_{AB} + V_{BA} - qL = 0 \Rightarrow V_{BA} = 0.521qL$$

#### Member BC

Free body diagram:



End moments:

$$(e12.11.3) \Rightarrow M_{BC} = \frac{4EI\theta_B}{L} + \frac{2EI\theta_C}{L} - \frac{6EI\Delta_B}{L^2} + \frac{qL^2}{12}$$

$$= 4(0.0781qL^2) + 2(-0.3281qL^2) + 6(-0.0399qL^2) + \frac{qL^2}{12} = -0.0208qL^2$$

$$(e12.11.4) \Rightarrow M_{CB} = \frac{2EI\theta_B}{L} + \frac{4EI\theta_C}{L} - \frac{6EI\Delta_B}{L^2} - \frac{qL^2}{12}$$

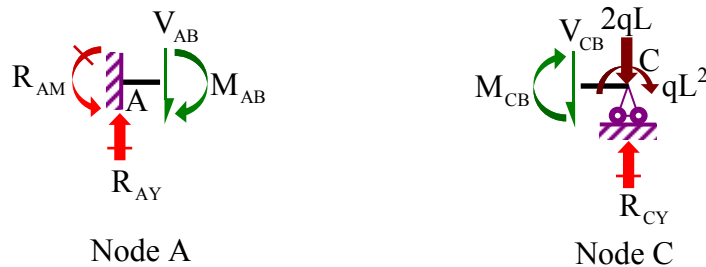
$$= 2(0.0781qL^2) + 4(-0.3281qL^2) - 6(-0.0399qL^2) - \frac{qL^2}{12} = -qL^2$$

End shear forces:

$$[\Sigma M_{\text{end } C} = 0] \Rightarrow V_{BC}(L) - M_{BC} - M_{CB} - qL(L/2) = 0 \Rightarrow V_{BC} = -0.521qL$$

$$[\Sigma F_{y, \text{ member } BC} = 0] \Rightarrow V_{BC} + V_{CB} - qL = 0 \Rightarrow V_{CB} = 1.521qL$$

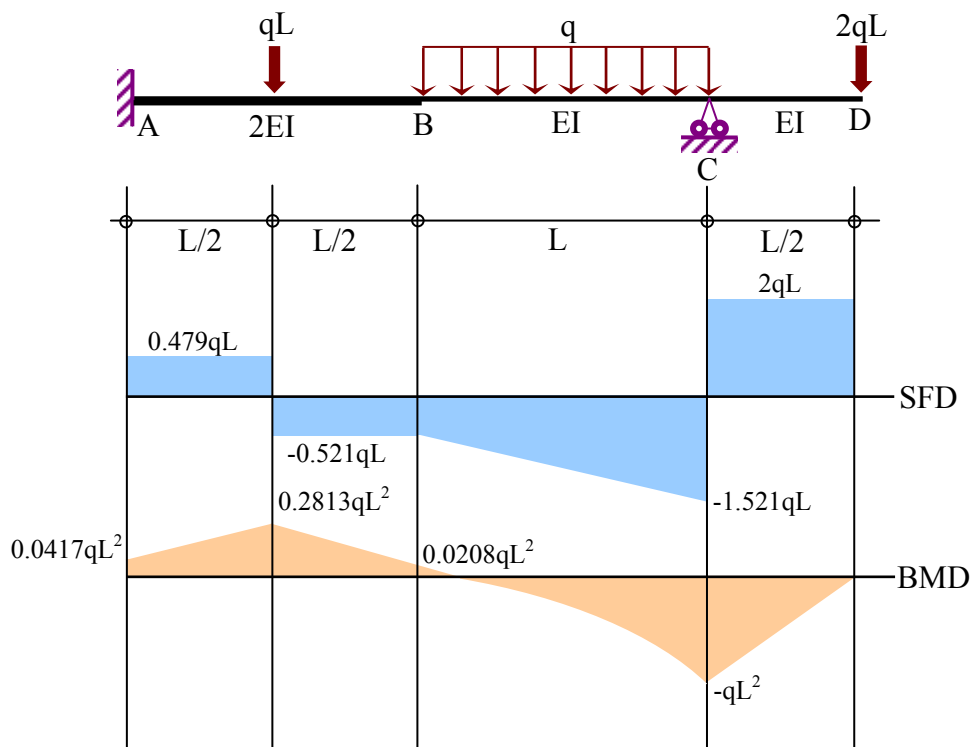
All support reactions can then be obtained from equilibrium of nodes B and C as follows:



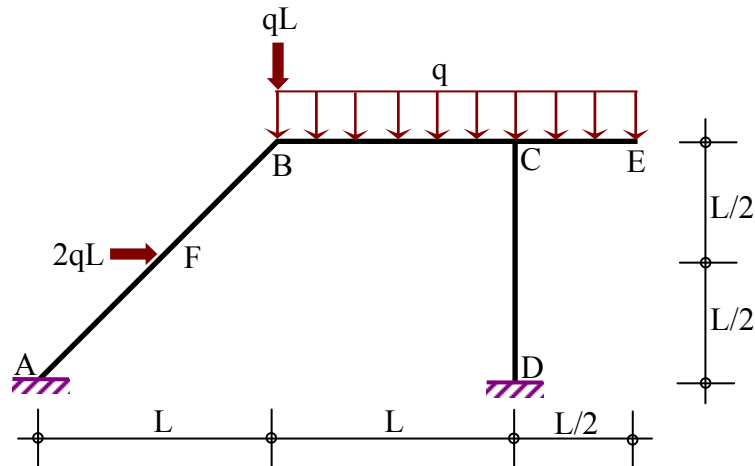
Node A:  $R_{AY} = V_{AB} = 0.479qL$  Upward ;  $R_{AM} = M_{AB} = -0.0417qL^2$  CCW

Node C:  $R_{CY} = V_{CB} + 2qL = 3.521qL$  Upward

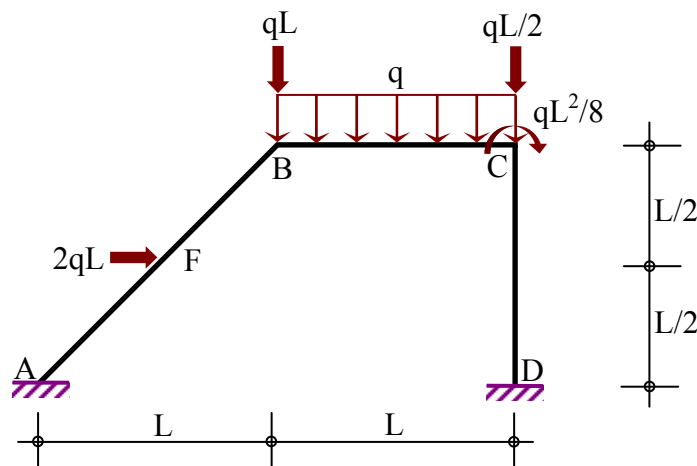
The shear force diagram (SFD) and bending moment diagram (BMD) of the beam are shown below



**Example 12.12** Use a method of slope-deflection equations to analyze a rigid frame under external loads shown below. The flexural rigidity of segments AB, BCE and CD are given by  $2EI$ ,  $EI$  and  $2EI$ , respectively.



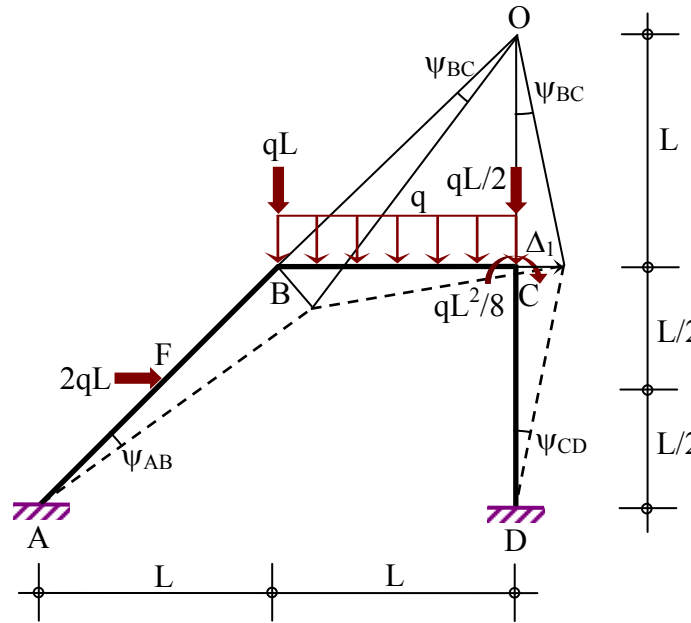
**Solution** Similar to the previous example, a statically determinate segment CE can be eliminated and replaced by a downward force  $qL/2$  and a clockwise moment  $qL^2/8$  acting at a point C as indicated below. Once the reduced structure shown below is analyzed, results for the segment CE can readily be included.



Now, let us discretize the reduced structure into three members (i.e. members AB, BC and CD) with four nodes (i.e. nodes A, B, C, and D). The number of rotational degrees of freedom is given by  $N_r = 4 + 0 - 2 = 2$  (with two unknown rotations  $\theta_B$  and  $\theta_C$ ) and the number of sway degrees of freedom is equal to  $N_s = 2(4) - 4 - 3 = 1$  (with the displacement at point C, denoted by  $\Delta_1$ , being chosen to represent this sway degree of freedom). Thus, the total number of primary unknowns is equal to  $2 + 1 = 3$  (with all primary unknowns in terms of  $\theta_B$ ,  $\theta_C$  and  $\Delta_1$ ). The sway pattern associated with the sway degree of freedom can be sketched as indicated below.

By using the ICR concept, the displacement at any point and the sway angle of all members can be expressed in terms of the sway degree of freedom  $\Delta_1$  as follows. Since a point D is a fixed point, it is the ICR of the member CD. By using the relation (12.68) along with the known ICR and known horizontal displacement at point C, the sway angle of the member CD is equal to  $\psi_{CD} = -\Delta_1/L$ . Since a point A is a fixed point, it is the ICR of the member AB. By using the fact that the movement of point B is perpendicular to the member AB and the movement of point C is

perpendicular to the member CD, the location of the ICR of the member BC can readily be obtained from the intersection of the line AB and the line CD, denoted by a point O. Again, by using the relation (12.68) along with the known ICR and known horizontal displacement at point C, the sway angle of the member BC is equal to  $\psi_{BC} = \Delta_1/L$ . Now, the horizontal displacement at point B can readily be obtained from  $\Delta_{BH} = \psi_{BC}(L) = \Delta_1$  (Rightward). Finally, by using the relation (12.68) along with the known ICR and known horizontal displacement at point B, the sway angle of the member AB is equal to  $\psi_{AB} = -\Delta_1/L$ .



The end rotations, the sway angle in terms of the sway degree of freedom  $\Delta_1$ , the fixed-end moments, and the slope-deflection equations for the members AB, BC and CD are given below.

**Member AB** For this member,  $\theta_A = 0$ ,  $\psi_{AB} = -\Delta_1/L$  and

$$FEM_{AB} = \frac{(2qL)L}{8} = \frac{qL^2}{4} \quad ; \quad FEM_{BA} = -\frac{(2qL)L}{8} = -\frac{qL^2}{4}$$

The slope-deflection equations become

$$M_{AB} = \frac{2(2EI)}{\sqrt{2}L} (2 \cdot 0 + \theta_B) - \frac{6(2EI)}{\sqrt{2}L} \left( -\frac{\Delta_1}{L} \right) + \frac{qL^2}{4} = \frac{2\sqrt{2}EI\theta_B}{L} + \frac{6\sqrt{2}EI\Delta_1}{L^2} + \frac{qL^2}{4} \quad (e12.12.1)$$

$$M_{BA} = \frac{2(2EI)}{\sqrt{2}L} (0 + 2\theta_B) - \frac{6(2EI)}{\sqrt{2}L} \left( -\frac{\Delta_1}{L} \right) - \frac{qL^2}{4} = \frac{4\sqrt{2}EI\theta_B}{L} + \frac{6\sqrt{2}EI\Delta_1}{L^2} - \frac{qL^2}{4} \quad (e12.12.2)$$

**Member BC** For this member,  $\psi_{BC} = \Delta_1/L$  and

$$FEM_{BC} = \frac{qL^2}{12} \quad ; \quad FEM_{CB} = -\frac{qL^2}{12}$$

The slope-deflection equations become

$$M_{BC} = \frac{2EI}{L} (2\theta_B + \theta_C) - \frac{6EI}{L} \left( \frac{\Delta_1}{L} \right) + \frac{qL^2}{12} = \frac{4EI\theta_B}{L} + \frac{2EI\theta_C}{L} - \frac{6EI\Delta_1}{L^2} + \frac{qL^2}{12} \quad (e12.12.3)$$

$$M_{CB} = \frac{2EI}{L}(\theta_B + 2\theta_C) - \frac{6EI}{L}\left(\frac{\Delta_1}{L}\right) - \frac{qL^2}{12} = \frac{2EI\theta_B}{L} + \frac{4EI\theta_C}{L} - \frac{6EI\Delta_1}{L^2} - \frac{qL^2}{12} \quad (\text{e12.12.4})$$

Member CD For this member,  $\theta_D = 0$ ,  $\psi_{CD} = -\Delta_1/L$ ,  $FEM_{CD} = FEM_{DC} = 0$ , and the slope-deflection equations become

$$M_{CD} = \frac{2(2EI)}{L}(2\theta_C + 0) - \frac{6(2EI)}{L}\left(-\frac{\Delta_1}{L}\right) + 0 = \frac{8EI\theta_C}{L} + \frac{12EI\Delta_1}{L^2} \quad (\text{e12.12.5})$$

$$M_{DC} = \frac{2(2EI)}{L}(\theta_C + 0) - \frac{6(2EI)}{L}\left(-\frac{\Delta_1}{L}\right) = \frac{4EI\theta_C}{L} + \frac{12EI\Delta_1}{L^2} \quad (\text{e12.12.6})$$

Two moment equilibrium equations associated with nodes B and C are obtained as follows.

Moment equilibrium at node B

$$\begin{aligned} [\Sigma M_{\text{node B}} = 0] &\Rightarrow M_{BA} + M_{BC} = 0 \\ &\Rightarrow \left( \frac{4\sqrt{2}EI\theta_B}{L} + \frac{6\sqrt{2}EI\Delta_1}{L^2} - \frac{qL^2}{4} \right) + \left( \frac{4EI\theta_B}{L} + \frac{2EI\theta_C}{L} - \frac{6EI\Delta_1}{L^2} + \frac{qL^2}{12} \right) = 0 \\ &\Rightarrow \frac{4(1+\sqrt{2})EI\theta_B}{L} + \frac{2EI\theta_C}{L} + \frac{6(\sqrt{2}-1)EI\Delta_1}{L^2} = \frac{qL^2}{6} \end{aligned} \quad (\text{e12.12.7})$$

Moment equilibrium at node C

$$\begin{aligned} [\Sigma M_{\text{node C}} = 0] &\Rightarrow M_{CB} + M_{CD} + qL^2/8 = 0 \\ &\Rightarrow \left( \frac{2EI\theta_B}{L} + \frac{4EI\theta_C}{L} - \frac{6EI\Delta_1}{L^2} - \frac{qL^2}{12} \right) + \left( \frac{8EI\theta_C}{L} + \frac{12EI\Delta_1}{L^2} \right) + \frac{qL^2}{8} = 0 \\ &\Rightarrow \frac{2EI\theta_B}{L} + \frac{12EI\theta_C}{L} + \frac{6EI\Delta_1}{L^2} = -\frac{qL^2}{24} \end{aligned} \quad (\text{e12.12.8})$$

The third equilibrium equation associated with the sway degree of freedom is obtained, as shown below, by using the principle of virtual work along with the sway pattern being chosen as a special choice of the virtual displacement.

$$\begin{aligned} \delta W_{\text{ext}} &= (2qL)|\psi_{AB}|(L/2) + (qL)|\psi_{BC}|(L) + (qL)|\psi_{BC}|(L/2) + (qL/2)|\psi_{BC}|(0) = 5qL\Delta_1/2 \\ \delta W_{\text{int}} &= -(M_{AB} + M_{BA})\psi_{AB} - (M_{BC} + M_{CB})\psi_{BC} - (M_{CD} + M_{DC})\psi_{CD} \\ &= (M_{AB} + M_{BA})\Delta_1/L - (M_{BC} + M_{CB})\Delta_1/L + (M_{CD} + M_{DC})\Delta_1/L \\ \delta W_{\text{ext}} = \delta W_{\text{int}} &\Rightarrow 5qL\Delta_1/2 = (M_{AB} + M_{BA})\Delta_1/L - (M_{BC} + M_{CB})\Delta_1/L + (M_{CD} + M_{DC})\Delta_1/L \\ &\Rightarrow M_{AB} + M_{BA} - M_{BC} - M_{CB} + M_{CD} + M_{DC} = 5qL^2/2 \\ &\Rightarrow \frac{6(\sqrt{2}-1)EI\theta_B}{L} + \frac{6EI\theta_C}{L} + \frac{12(3+\sqrt{2})EI\Delta_1}{L^2} = \frac{5qL^2}{2} \end{aligned} \quad (\text{e12.12.9})$$

By solving a system of three linear equations (e12.12.7), (e12.12.8) and (e12.12.9), we obtain

$$\frac{EI\theta_B}{L} = 0.0106qL^2 \quad (\text{e12.12.10})$$

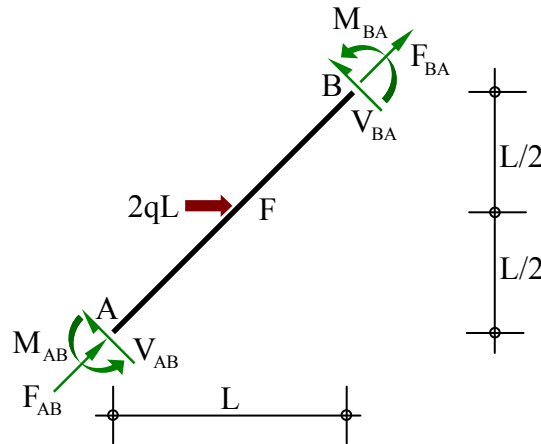
$$\frac{EI\theta_C}{L} = -0.0303qL^2 \quad (\text{e12.12.11})$$

$$\frac{EI\Delta_1}{L^2} = 0.0501qL^2 \quad (\text{e12.12.12})$$

Once all primary unknowns are solved, the end moments of members AB, BC and CD are obtained from the slope-deflection equations (e12.12.1)-(e12.12.6) and the end shear forces can be computed from equilibrium of members as shown below. In addition, relations between the end axial forces are also obtained from equilibrium of forces along the axis of each member.

#### Member AB

Free body diagram:



End moments:

$$(\text{e12.12.1}) \Rightarrow M_{AB} = \frac{2\sqrt{2}EI\theta_B}{L} + \frac{6\sqrt{2}EI\Delta_1}{L^2} + \frac{qL^2}{4} = 2\sqrt{2}(0.0106qL^2) + 6\sqrt{2}(0.0501qL^2) + \frac{qL^2}{4} = 0.706qL^2$$

$$(\text{e12.12.2}) \Rightarrow M_{BA} = \frac{4\sqrt{2}EI\theta_B}{L} + \frac{6\sqrt{2}EI\Delta_1}{L^2} - \frac{qL^2}{4} = 4\sqrt{2}(0.0106qL^2) + 6\sqrt{2}(0.0501qL^2) - \frac{qL^2}{4} = 0.236qL^2$$

End shear forces:

$$[\Sigma M_{\text{end B}} = 0] \Rightarrow V_{AB}(\sqrt{2}L) - M_{AB} - M_{BA} - 2qL(L/2) = 0 \Rightarrow V_{AB} = 1.373qL$$

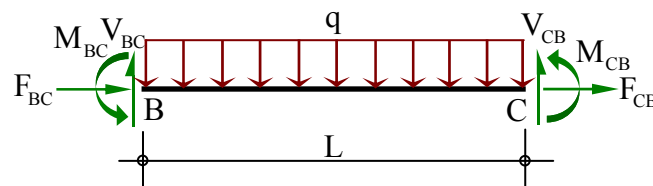
$$[\Sigma F_{y, \text{ member AB}} = 0] \Rightarrow V_{AB} + V_{BA} - 2qL/\sqrt{2} = 0 \Rightarrow V_{BA} = 0.042qL$$

End axial forces:

$$[\Sigma F_{x, \text{ member AB}} = 0] \Rightarrow F_{AB} + F_{BA} + 2qL/\sqrt{2} = 0 \quad (\text{e12.12.13})$$

#### Member BC

Free body diagram:



End moments:

$$(\text{e12.12.3}) \Rightarrow M_{BC} = \frac{4EI\theta_B}{L} + \frac{2EI\theta_C}{L} - \frac{6EI\Delta_1}{L^2} + \frac{qL^2}{12}$$

$$= 4(0.0106qL^2) + 2(-0.0303qL^2) - 6(0.0501qL^2) + \frac{qL^2}{12} = -0.236qL^2$$

$$(e12.12.4) \Rightarrow M_{CB} = \frac{2EI\theta_B}{L} + \frac{4EI\theta_C}{L} - \frac{6EI\Delta_1}{L^2} - \frac{qL^2}{12}$$

$$= 2(0.0106qL^2) + 4(-0.0303qL^2) - 6(0.0501qL^2) - \frac{qL^2}{12} = -0.484qL^2$$

End shear forces:

$$[\Sigma M_{\text{end C}} = 0] \Rightarrow V_{BC}(L) - M_{BC} - M_{CB} - qL(L/2) = 0 \Rightarrow V_{BC} = -0.220qL$$

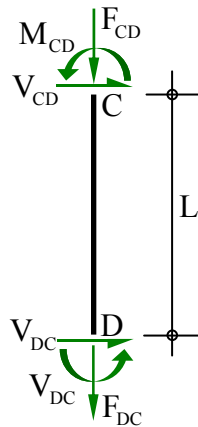
$$[\Sigma F_{y, \text{ member BC}} = 0] \Rightarrow V_{BC} + V_{CB} - qL = 0 \Rightarrow V_{CB} = 1.220qL$$

End axial forces:

$$[\Sigma F_{x, \text{ member BC}} = 0] \Rightarrow F_{BC} + F_{CB} = 0 \quad (e12.12.14)$$

### Member CD

Free body diagram:



End moments:

$$(e12.12.5) \Rightarrow M_{CD} = \frac{8EI\theta_C}{L} + \frac{12EI\Delta_1}{L^2} = 8(-0.0303qL^2) + 12(0.0501qL^2) = 0.359qL^2$$

$$(e12.12.6) \Rightarrow M_{DC} = \frac{4EI\theta_C}{L} + \frac{12EI\Delta_1}{L^2} = 4(-0.0303qL^2) + 12(0.0501qL^2) = 0.480qL^2$$

End shear forces:

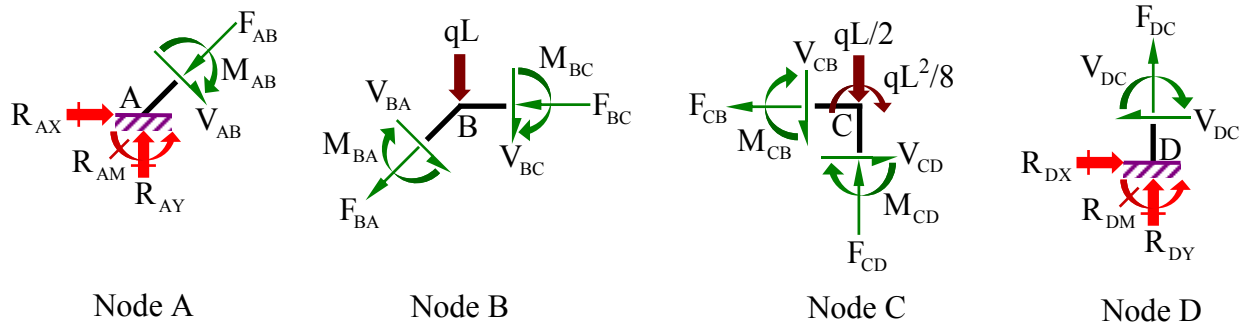
$$[\Sigma M_{\text{end D}} = 0] \Rightarrow V_{CD}(L) - M_{BC} - M_{CB} = 0 \Rightarrow V_{CD} = 0.839qL$$

$$[\Sigma F_{y, \text{ member CD}} = 0] \Rightarrow V_{CD} + V_{DC} = 0 \Rightarrow V_{DC} = -0.839qL$$

End axial forces:

$$[\Sigma F_{x, \text{ member CD}} = 0] \Rightarrow F_{CD} + F_{DC} = 0 \quad (e12.12.15)$$

The support reactions and end axial forces are obtained by considering equilibrium of nodes A, B, C and D as shown below.



Node C:  $F_{CB} = V_{CD} = -0.839qL$  ;  $F_{CD} = V_{CB} + qL/2 = 1.720qL$

(e12.12.14)  $\Rightarrow F_{BC} = -F_{CB} = 0.839qL$  and (e12.12.15)  $\Rightarrow F_{DC} = -F_{CD} = -1.720qL$

Node B:  $F_{BA} = -F_{BC}/\sqrt{2} - V_{BC}/\sqrt{2} - qL/\sqrt{2} = -1.145qL$  ;

(e12.12.13)  $\Rightarrow F_{AB} = -F_{BA} - 2qL/\sqrt{2} = -0.269qL$

Node A:  $R_{AX} = F_{AB}/\sqrt{2} - V_{AB}/\sqrt{2} = -1.161qL$  Leftward ;

$R_{AY} = F_{AB}/\sqrt{2} + V_{AB}/\sqrt{2} = 0.780qL$  Upward ;

$R_{AM} = M_{AB} = 0.706qL^2$  CCW

Node D:  $R_{DX} = V_{DC} = -0.839qL$  Leftward ;

$R_{DY} = -F_{DC} = 1.720qL$  Upward ;

$R_{DM} = M_{DC} = 0.480qL^2$  CCW

The AFD, SFD and BMD for members AB, BC, and BC can also be obtained using standard procedure.

## 12.7 Treatment of Special Structures

In this section, we demonstrate the applications of the slope-deflection method to certain classes of structures such as structures containing hinges and structures possessing symmetric geometry and subjected either to symmetric or anti-symmetric loading conditions. These special structures hold several attractive features that, when properly utilized, allow the significant reduction of the analysis effort; for instance, the number of primary unknowns and governing equilibrium equations can substantially be reduced. The key component that contributes to such reduction is a set of modified slope-deflection equations discussed in section 12.5. Several examples are presented further below to clearly demonstrate the application of such modified equations.

### 12.7.1 Structure containing points of prescribed bending moment

Here, we focus attention on structures that contain points where the bending moment is prescribed or known a priori. For instance, a beam shown in Figure 12.25(a) contains points A and D where the bending moment is known (i.e.  $M_A = 0$  and  $M_D = M_0$ ) and a gable frame shown in Figure 12.25(b) contains an interior hinge at a point C (i.e.  $M_C = 0$ ) and a point E where the bending moment vanishes (i.e.  $M_E = 0$ ). As clearly discussed in the previous section, an interior hinge cannot appear within the member in the discretization process and presence of such internal moment release at nodes generally introduces additional independent rotational degrees of freedom.



For instance, at the point C of a gable frame shown in Figure 12.25(a), there are two independent rotational degrees of freedom (i.e.  $\theta_{CL}$  and  $\theta_{CR}$ ), one associated with the rotation at a point just the left of the point C and the other corresponding to the rotation at a point just to the right of the point C.

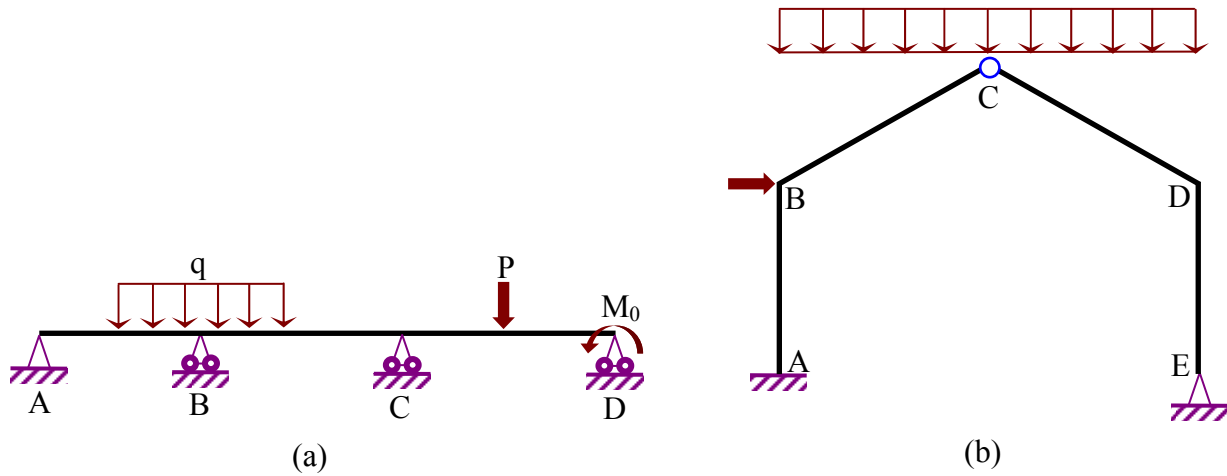


Figure 12.25 Schematic of structures containing points where bending moment is known a priori: (a) continuous beam and (b) gable frame.

Now, let PQ be a generic member in the discretized structure that the end moment  $M_{QP}$  is prescribed or known a priori, i.e.  $M_{QP} = M_{Q0}$  where  $M_{Q0}$  is a known moment. Following the same procedure as discussed in subsection 12.5.1, the rotation at the end Q (where the end moment is known) can be obtained in terms of the rotation  $\theta_P$ , the sway angle  $\psi_{PQ}$ , the fixed-end moment  $FEM_{QP}$ , and the prescribed moment  $M_{Q0}$  by using the slope-deflection equation at the end Q as follows:

$$M_{QP} = \frac{2EI}{L}(2\theta_Q + \theta_P) - \frac{6EI}{L}\psi_{PQ} + FEM_{QP} = M_{Q0} \Rightarrow \theta_Q = -\frac{1}{2}\theta_P + \frac{3}{2}\psi_{PQ} - \frac{L}{4EI}(FEM_{QP} - M_{Q0}) \quad (12.73)$$

It should be noted that the relation (12.73) can be employed to compute the rotation at the end Q once the rotation at the end P and the sway angle  $\psi_{PQ}$  are solved. By inserting the rotation  $\theta_Q$  given by (12.73) into the slope-deflection equation at the end P, such equation now becomes

$$M_{PQ} = \frac{3EI}{L}\theta_P - \frac{3EI}{L}\psi_{PQ} + \overline{FEM}_{PQ} \quad (12.74)$$

where  $\overline{FEM}_{PQ}$  is the modified fixed-end moment at the end P given by

$$\overline{FEM}_{PQ} = FEM_{PQ} - \frac{1}{2}FEM_{QP} + \frac{1}{2}M_{Q0} \quad (12.75)$$

By following the same procedure, we can also obtain the modified slope-deflection equation at the end Q, for the case that the end moment  $M_{PQ}$  is prescribed equal to  $M_{P0}$ , in a form

$$M_{QP} = \frac{3EI}{L}\theta_Q - \frac{3EI}{L}\psi_{PQ} + \overline{FEM}_{QP} \quad (12.76)$$

where  $\overline{\text{FEM}}_{QP}$  is the modified fixed-end moment at the end Q given by

$$\overline{\text{FEM}}_{QP} = \text{FEM}_{QP} - \frac{1}{2}\text{FEM}_{PQ} + \frac{1}{2}M_{P0} \quad (12.77)$$

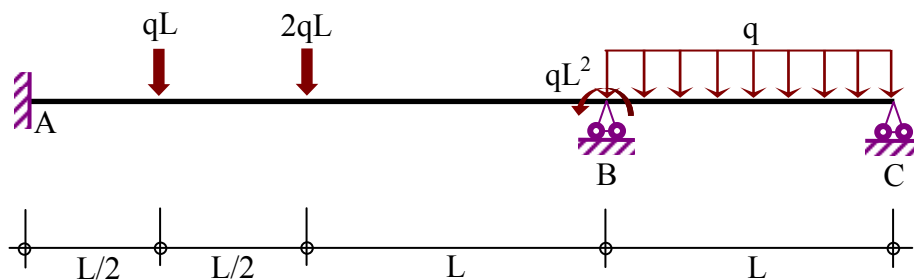
Once the rotation at the end Q and the sway angle  $\psi_{PQ}$  are solved, the rotation at the end P can be computed from

$$\theta_P = -\frac{1}{2}\theta_Q + \frac{3}{2}\psi_{PQ} - \frac{L}{4EI}(\text{FEM}_{PQ} - M_{P0}) \quad (12.78)$$

Note that a condition associated with the prescribed moment at one end was utilized not only to reduce the number of slope-deflection equations from two to one but also to eliminate the rotation at that end from the modified slope-deflection equation (12.74) or (12.76). This positive feature is useful, when applied to a discretized structure that contains members with the prescribed end moment, to reduce the number of rotational degrees of freedom. Specifically, the rotations at nodes where the bending moment are prescribed are not treated as primary unknowns and, as a result, there is no need to set up moment equilibrium at those nodes.

For instance, the number of rotational degrees of freedom of the continuous beam shown in Figure 12.25(a) can be reduced from 4 to 2 if the modified slope-deflection equation is utilized for the members AB and CD. In particular, the rotations  $\theta_A$  and  $\theta_D$  are not treated as primary unknowns and there is no need to establish moment equilibrium at nodes A and D. Similarly, if the modified slope-deflection equation is utilized for the members BC, CD and DE, the number of rotational degrees of freedom is reduced from 6 to 3. The rotations  $\theta_{CL}$ ,  $\theta_{CR}$  and  $\theta_E$  are not treated as primary unknowns and there is no need to establish moment equilibrium at nodes C and E. Two examples considered below clearly demonstrate the application of the modified slope-deflection equation for a member with prescribed end moment.

**Example 12.13** Use the slope-deflection method along with the modified slope-deflection equation to resolve the continuous beam shown in Example 12.9 for the case of no support settlement.



**Solution** For a given beam, the bending moment at a point C is known a priori, i.e.  $M_C = 0$ . By following the same discretization as that utilized in Example 12.9 (i.e. the discretized beam consists of three nodes A, B, and C and two members AB and BC) along with using the modified slope-deflection equation for the member BC, the number of rotational degrees of freedom reduces to 1 (i.e.  $\theta_B$ ) whereas the number of sway degrees of freedom is still 0. Therefore, the number of primary unknowns is equal to  $1 + 0 = 1$ .

The end rotations, the sway angles, the fixed-end moments, and the slope-deflection equations for the members AB and BC are shown below.

Member AB The slope-deflection equations for this member are the same as those given in Example 12.9, i.e.

$$M_{AB} = \frac{EI\theta_B}{L} + \frac{25qL^2}{32} \quad (\text{e12.13.1})$$

$$M_{BA} = \frac{2EI\theta_B}{L} - \frac{19qL^2}{32} \quad (\text{e12.13.2})$$

Member BC For this member,  $M_{C0} = 0$ ,  $\psi_{BC} = 0$ ,  $FEM_{BC} = qL^2/12$ ,  $FEM_{CB} = -qL^2/12$  and the modified fixed-end moment at the end B is given by

$$\overline{FEM}_{BC} = FEM_{BC} - \frac{1}{2}FEM_{CB} + \frac{1}{2}M_{C0} = \frac{qL^2}{12} - \frac{1}{2}\left(-\frac{qL^2}{12}\right) + \frac{1}{2}(0) = \frac{qL^2}{8}$$

The modified slope-deflection equations become

$$M_{BC} = \frac{3EI}{L}\theta_B - \frac{3EI}{L}(0) + \frac{qL^2}{8} = \frac{3EI\theta_B}{L} + \frac{qL^2}{8} \quad (\text{e12.13.3})$$

The rotation at the end C is given by

$$\theta_C = -\frac{1}{2}\theta_B + \frac{3}{2}(0) - \frac{L}{4EI}\left(-\frac{qL^2}{12} - 0\right) = -\frac{1}{2}\theta_B + \frac{qL^3}{48EI} \quad (\text{e12.13.4})$$

For this particular case, it is required to set up only one equilibrium equation associated with the moment equilibrium at node B and this can be achieved as demonstrated below.

Moment equilibrium at node B

$$\begin{aligned} [\Sigma M_{\text{node B}} = 0] &\Rightarrow M_{BA} + M_{BC} - qL^2 = 0 \\ &\Rightarrow \left(\frac{2EI\theta_B}{L} - \frac{19qL^2}{32}\right) + \left(\frac{3EI\theta_B}{L} + \frac{qL^2}{8}\right) - qL^2 = 0 \\ &\Rightarrow \frac{5EI\theta_B}{L} = \frac{47qL^2}{32} \end{aligned} \quad (\text{e12.13.5})$$

By solving equation (e12.13.5), it yields

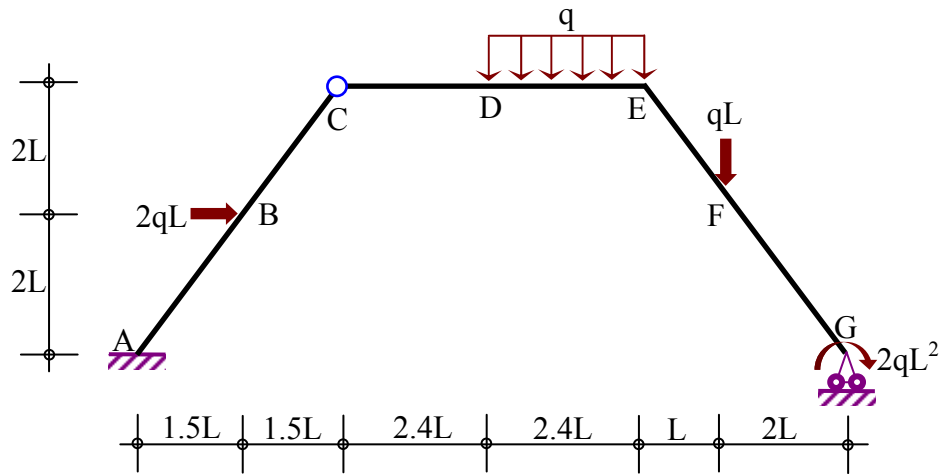
$$\frac{EI\theta_B}{L} = 0.2938qL^2 \quad (\text{e12.13.6})$$

The rotation at the end C can now be computed from (e12.13.4) and the result is given by

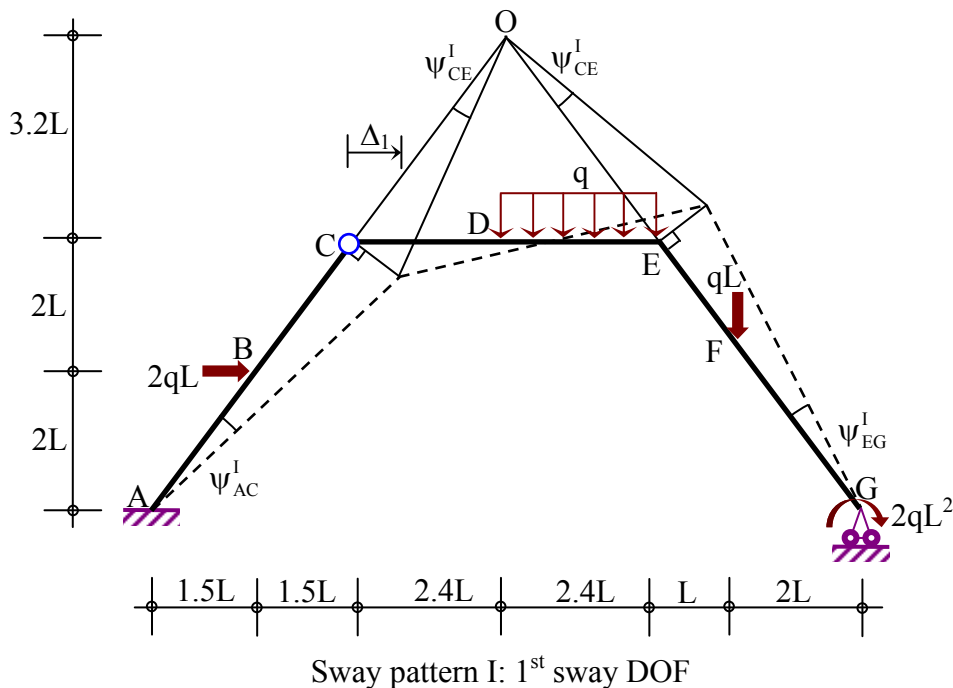
$$\theta_C = -\frac{1}{2}\theta_B + \frac{qL^3}{48EI} = -\frac{1}{2}(0.2938\frac{qL^3}{EI}) + \frac{qL^3}{48EI} = -0.1260\frac{qL^3}{EI} \quad (\text{e12.13.7})$$

Now, the end moments of the member AB can be obtained by inserting the rotation  $\theta_B$  into equations (e12.13.1) and (e12.13.2) and the moment at the end B of the member BC can be obtained from equation (e12.13.3); the moment at the end C is already known a priori, i.e.  $M_{CB} = M_{C0} = 0$ . The end shear forces, support reactions, SFD and BMD can also be obtained in the same fashion as those shown in Example 12.9.

**Example 12.14** Use the slope-deflection method along with the modified slope-deflection equation to analyze a rigid frame subjected to external loads shown below. The flexural rigidity  $EI$  is assumed to be constant throughout the structure.

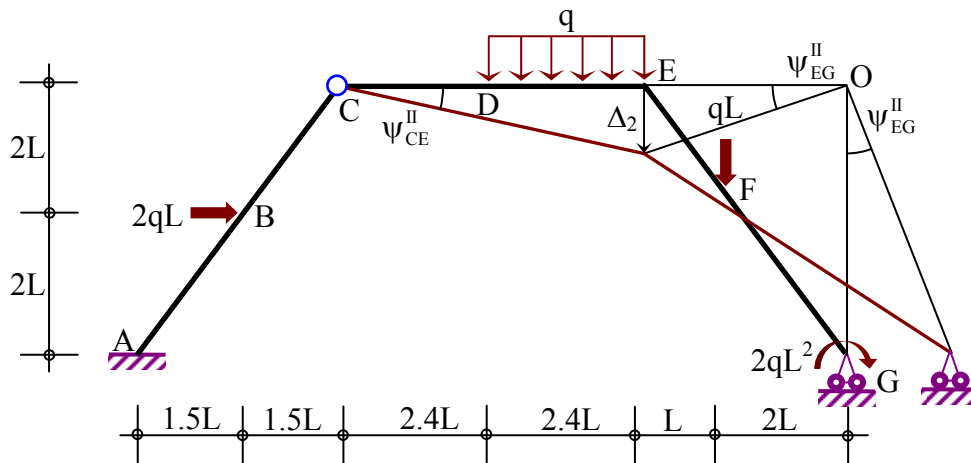


**Solution** To minimize the number of primary unknowns, let us discretize the structure into three members (i.e. members AC, CE and EG) with four nodes (i.e. nodes A, C, E and G) and. Based on this discretization, the number of rotational degrees of freedom is equal to  $N_r = 4 + 1 - 1 = 4$  (i.e.  $\theta_{CL}$ ,  $\theta_{CR}$ ,  $\theta_E$  and  $\theta_G$ ) and the number of sway degrees of freedom is equal to  $N_s = 2(4) - 3 - 3 = 2$ ; thus, the number of primary unknowns is equal to  $4 + 2 = 6$ . It is noted, however, that the bending moment at the hinge C vanishes and the bending moment at point G is known a priori. Based on this information, the modified slope-deflection equation can be applied to members AC, CE and EG and, as a result, the number of rotational degrees of freedom can be reduced from 4 to 1 (i.e. only  $\theta_E$  is treated as a rotational degree of freedom) and the number of primary unknowns can be reduced from 6 to 3.



Sway patterns associated with the two independent sway degrees of freedom of the discretized structure are sketched as shown in the figure below. The first sway pattern is associated

with the side-sway of the frame without the movement of the point G. By choosing the horizontal displacement at point C, denoted by  $\Delta_1$ , to represent the first sway degree of freedom, the sway angles of all members for this sway mode can be obtained in terms of  $\Delta_1$  using the ICR concept as demonstrated below. Since a point A is a fixed point, it is the ICR of the member AC. By using the relation (12.68) along with the known ICR and known horizontal displacement at point C, the sway angle of the member AC is equal to  $\psi_{AC}^I = -\Delta_1/4L$  (CW). Since a point G does not move for this sway mode, it is therefore the ICR of the member EG. By using the fact that the movement of point C is perpendicular to the member AC and the movement of point E is perpendicular to the member EG, the location of the ICR of the member CE can readily be obtained from the intersection of the line AC and the line EG, denoted by a point O. Again, by using the relation (12.68) along with the known ICR and known horizontal displacement at point C, the sway angle of the member CE is equal to  $\psi_{CE}^I = \Delta_1/3.2L$  (CCW). Now, the horizontal displacement at point E can readily be obtained from  $\Delta_{EH} = \psi_{CE}^I (3.2L) = \Delta_1$  (Rightward). Finally, by using the relation (12.68) along with the known ICR and known horizontal displacement at point E, the sway angle of the member EG is equal to  $\psi_{EG}^I = -\Delta_1/4L$ .



Sway pattern II: 2<sup>nd</sup> sway DOF

The second sway pattern is associated with the side-sway of the frame without the movement of the point C. As a result of the difference of the displacements at point C, this sway degree of freedom is obviously independent of the first sway mode. By choosing the vertical displacement at point E, denoted by  $\Delta_2$ , to represent the second sway degree of freedom, the sway angles of all members for this sway mode can be obtained in terms of  $\Delta_2$  by using the ICR concept. Since the member AC does not move, its sway angle vanishes (i.e.  $\psi_{AC}^{II} = 0$ ). Since a point C does not move, it is the ICR of the member CE. By using the relation (12.68) along with the known ICR and known vertical displacement at point E, the sway angle of the member CE is equal to  $\psi_{CE}^{II} = -\Delta_2/4.8L$  (CW). By using the fact that the movement of point E is perpendicular to the member CE and the point G can only move in the horizontal direction due to the roller support, the location of the ICR of the member EG can readily be obtained from the intersection of the line CE and the vertical line emanating from the point G, denoted by a point O. Again, by using the relation (12.68) along with the known ICR and known vertical displacement at point E, the sway angle of the member EG is equal to  $\psi_{EG}^{II} = \Delta_2/3L$  (CCW).

The sway angles associated with each sway degree of freedom and the total sway angles of the members AC, CE, and EG are summarized in the table below.

| Member | Sway angle due to 1 <sup>st</sup><br>sway DOF | Sway angle due to 2 <sup>nd</sup><br>sway DOF | Total sway angle                |
|--------|-----------------------------------------------|-----------------------------------------------|---------------------------------|
| AC     | $-\Delta_1/4L$                                | 0                                             | $-\Delta_1/4L$                  |
| CE     | $\Delta_1/3.2L$                               | $-\Delta_2/4.8L$                              | $\Delta_1/3.2L - \Delta_2/4.8L$ |
| EG     | $-\Delta_1/4L$                                | $\Delta_2/3L$                                 | $-\Delta_1/4L + \Delta_2/3L$    |

The end rotations, the sway angles, the fixed-end moments, and the slope-deflection equations for the members AC, CE and EG are shown below.

Member AC For this member,  $\theta_A = 0$ ,  $\psi_{AC} = -\Delta_1/4L$ ,  $M_{C0} = 0$ ,

$$FEM_{AC} = \frac{(2qL)(4L)}{8} = qL^2 \quad ; \quad FEM_{CA} = -\frac{(2qL)(4L)}{8} = -qL^2$$

and the modified fixed-end moment at the end A is given by

$$\overline{FEM}_{AC} = FEM_{AC} - \frac{1}{2}FEM_{CA} + \frac{1}{2}M_{C0} = qL^2 - \frac{1}{2}(-qL^2) + \frac{1}{2}(0) = \frac{3qL^2}{2}$$

The modified slope-deflection equations become

$$M_{AC} = \frac{3EI}{5L}(0) - \frac{3EI}{5L}\left(-\frac{\Delta_1}{4L}\right) + \frac{3qL^2}{2} = \frac{3EI\Delta_1}{20L^2} + \frac{3qL^2}{2} \quad (e12.14.1)$$

The rotation at the end C is given by

$$\theta_{CL} = -\frac{1}{2}(0) + \frac{3}{2}\left(-\frac{\Delta_1}{4L}\right) - \frac{5L}{4EI}(-qL^2 - 0) = -\frac{3\Delta_1}{8L} + \frac{5qL^3}{4EI} \quad (e12.14.2)$$

Member CE For this member,  $\psi_{CE} = \Delta_1/3.2L - \Delta_2/4.8L$ ,  $M_{C0} = 0$ ,

$$FEM_{CE} = \frac{q(4.8L)^2}{12} \left[ 4\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^4 \right] = \frac{3qL^2}{5}$$

$$FEM_{EC} = -\frac{q(4.8L)^2}{12} \left[ 6\left(\frac{1}{2}\right)^2 - 8\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^4 \right] = -\frac{33qL^2}{25}$$

and the modified fixed-end moment at the end E is given by

$$\overline{FEM}_{EC} = FEM_{EC} - \frac{1}{2}FEM_{CE} + \frac{1}{2}M_{C0} = -\frac{33qL^2}{25} - \frac{1}{2}\left(\frac{3qL^2}{5}\right) + \frac{1}{2}(0) = -\frac{81qL^2}{50}$$

The modified slope-deflection equations become

$$M_{EC} = \frac{3EI}{4.8L}\theta_E - \frac{3EI}{4.8L}\left(\frac{\Delta_1}{3.2L} - \frac{\Delta_2}{4.8L}\right) - \frac{81qL^2}{50} = \frac{5EI\theta_E}{8L} - \frac{25EI\Delta_1}{128L^2} + \frac{25EI\Delta_2}{192L^2} - \frac{81qL^2}{50} \quad (e12.14.3)$$

The rotation at the end C is given by

$$\theta_{CR} = -\frac{1}{2}\theta_E + \frac{3}{2}\left(\frac{\Delta_1}{3.2L} - \frac{\Delta_2}{4.8L}\right) - \frac{4.8L}{4EI}\left(\frac{3qL^2}{5} - 0\right) = -\frac{1}{2}\theta_E + \frac{15\Delta_1}{32L} - \frac{5\Delta_2}{16L} - \frac{18qL^3}{25EI} \quad (e12.14.4)$$

**Member EG** For this member,  $\psi_{EG} = -\Delta_1/4L + \Delta_2/3L$ ,  $M_{C0} = -2qL^2$ ,

$$FEM_{EG} = \frac{(qL)(2L)^2(L)}{(3L)^2} = \frac{4qL^2}{9} \quad ; \quad FEM_{GE} = -\frac{(qL)(L)^2(2L)}{(3L)^2} = -\frac{2qL^2}{9}$$

and the modified fixed-end moment at the end E is given by

$$\overline{FEM}_{EG} = FEM_{EG} - \frac{1}{2}FEM_{GE} + \frac{1}{2}M_{G0} = \frac{4qL^2}{9} - \frac{1}{2}\left(-\frac{2qL^2}{9}\right) + \frac{1}{2}(-2qL^2) = -\frac{4qL^2}{9}$$

The modified slope-deflection equations become

$$M_{EG} = \frac{3EI}{5L}\theta_E - \frac{3EI}{5L}\left(-\frac{\Delta_1}{4L} + \frac{\Delta_2}{3L}\right) - \frac{4qL^2}{9} = \frac{3EI\theta_E}{5L} + \frac{3EI\Delta_1}{20L^2} - \frac{EI\Delta_2}{5L^2} - \frac{4qL^2}{9} \quad (e12.14.5)$$

The rotation at the end G is given by

$$\theta_G = -\frac{1}{2}\theta_E + \frac{3}{2}\left(-\frac{\Delta_1}{4L} + \frac{\Delta_2}{3L}\right) - \frac{5L}{4EI}\left(-\frac{2qL^2}{9} + 2qL^2\right) = -\frac{1}{2}\theta_E - \frac{3\Delta_1}{8L} + \frac{\Delta_2}{2L} - \frac{20qL^3}{9EI} \quad (e12.14.6)$$

Next, we set up three equilibrium equations, one associated with the moment equilibrium of node E and the other two corresponding to equilibrium equations of the two sway modes. The first equation is obtained as follows.

**Moment equilibrium at node E**

$$\begin{aligned} [\Sigma M_{\text{node E}} = 0] &\Rightarrow M_{EC} + M_{EG} = 0 \\ &\Rightarrow \left(\frac{5EI\theta_E}{8L} - \frac{25EI\Delta_1}{128L^2} + \frac{25EI\Delta_2}{192L^2} - \frac{81qL^2}{50}\right) + \left(\frac{3EI\theta_E}{5L} + \frac{3EI\Delta_1}{20L^2} - \frac{EI\Delta_2}{5L^2} - \frac{4qL^2}{9}\right) = 0 \\ &\Rightarrow \frac{49EI\theta_E}{40L} - \frac{29EI\Delta_1}{640L^2} - \frac{67EI\Delta_2}{960L^2} = \frac{929qL^2}{450} \end{aligned} \quad (e12.14.7)$$

The second equilibrium equation (associated with the first sway degree of freedom) is obtained by applying the principle of virtual work along with the sway pattern I being chosen as the virtual displacement. Results are shown below.

$$\begin{aligned} \delta W_{\text{ext}} &= (2qL)\psi_{AC}^I(2L) - (2.4qL)\psi_{CE}^I(1.2L) - (qL)\psi_{EG}^I(2L) = -2qL\Delta_1/5 \\ \delta W_{\text{int}} &= -(M_{AC} + M_{CA})\psi_{AC}^I - (M_{CE} + M_{EC})\psi_{CE}^I - (M_{EG} + M_{GE})\psi_{EG}^I \\ &= (M_{AC} + 0)\Delta_1/4L - (0 + M_{EC})\Delta_1/3.2L + (M_{EG} - 2qL^2)\Delta_1/4L \\ \delta W_{\text{ext}} = \delta W_{\text{int}} &\Rightarrow -2qL\Delta_1/5 = M_{AC}(\Delta_1/4L) - M_{EC}(\Delta_1/3.2L) + (M_{EG} - 2qL^2)\Delta_1/4L \\ &\Rightarrow M_{AC}/4 - M_{EC}/3.2 + M_{EG}/4 = qL^2/10 \\ &\Rightarrow -\frac{29EI\theta_E}{640L} + \frac{1393EI\Delta_1}{10240L^2} - \frac{1393EI\Delta_2}{15360L^2} = -\frac{193qL^2}{288} \end{aligned} \quad (e12.14.8)$$

The third equilibrium equation (associated with the second sway degree of freedom) is obtained by applying the principle of virtual work along with the sway pattern II being chosen as the virtual displacement. Results are shown below.

$$\begin{aligned} \delta W_{\text{ext}} &= (2qL)\psi_{AC}^{II}(2L) + (2.4qL)\psi_{CE}^{II}(3.6L) + (qL)\psi_{EG}^{II}(2L) = 37qL\Delta_2/15 \\ \delta W_{\text{int}} &= -(M_{AC} + M_{CA})\psi_{AC}^{II} - (M_{CE} + M_{EC})\psi_{CE}^{II} - (M_{EG} + M_{GE})\psi_{EG}^{II} \end{aligned}$$

$$\begin{aligned}
 &= (M_{AC} + 0)(0) + (0 + M_{EC})\Delta_2 / 4.8L - (M_{EG} - 2qL^2)\Delta_2 / 3L \\
 \delta W_{\text{ext}} = \delta W_{\text{int}} &\Rightarrow 37qL\Delta_2 / 15 = M_{EC}(\Delta_2 / 4.8L) - (M_{EG} - 2qL^2)\Delta_2 / 3L \\
 &\Rightarrow M_{EC} / 4.8 - M_{EG} / 3 = 9qL^2 / 5 \\
 &\Rightarrow -\frac{67EI\theta_E}{960L} - \frac{1393EI\Delta_1}{15360L^2} + \frac{2161EI\Delta_2}{23040L^2} = \frac{4297qL^2}{2160}
 \end{aligned} \tag{e12.14.9}$$

By solving a system of three linear equations (e12.14.7), (e12.14.8) and (e12.14.9) simultaneously, we obtain

$$\frac{EI\theta_E}{L} = 7.110qL^2 \tag{e12.14.10}$$

$$\frac{EI\Delta_1}{L^2} = 42.513qL^2 \tag{e12.14.11}$$

$$\frac{EI\Delta_2}{L^2} = 67.606qL^2 \tag{e12.14.12}$$

The rotation at the hinge C (i.e.  $\theta_{CL}$  and  $\theta_{CR}$ ) and the rotation at the point G (i.e.  $\theta_G$ ) can readily be computed from (e12.14.2), (e12.14.4) and (e12.14.6), respectively, and results are given by

$$\begin{aligned}
 \theta_{CL} &= -\frac{3\Delta_1}{8L} + \frac{5qL^3}{4EI} = -14.692 \frac{qL^3}{EI} \\
 \theta_{CR} &= -\frac{1}{2}\theta_E + \frac{15\Delta_1}{32L} - \frac{5\Delta_2}{16L} - \frac{18qL^3}{25EI} = -5.438 \frac{qL^3}{EI} \\
 \theta_G &= -\frac{1}{2}\theta_E - \frac{3\Delta_1}{8L} + \frac{\Delta_2}{2L} - \frac{20qL^3}{9EI} = 11.842 \frac{qL^3}{EI}
 \end{aligned}$$

Once all primary unknowns are solved, the end moments and end shear forces of all members, all support reactions, and the AFD, SFD and BMD can readily be obtained.

## 12.7.2 Geometrically symmetric structures

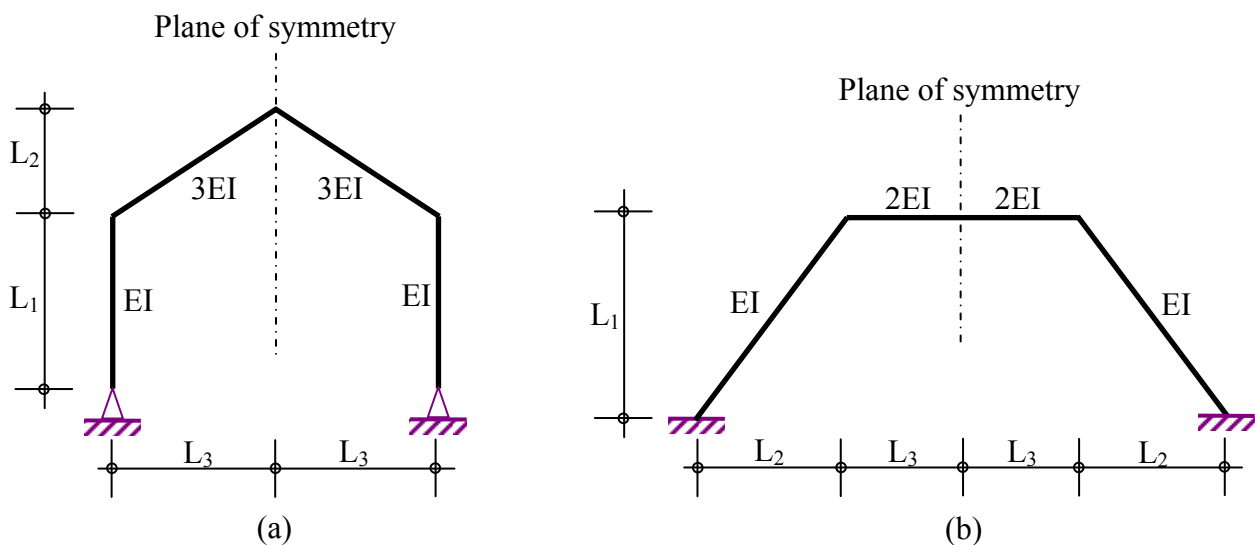


Figure 12.26 Schematic of geometrically symmetric structures with plane of symmetry passing through: (a) joint and (b) mid-point of segment



Another special type of structures that allows the reduction of the number of primary unknowns and also the computational effort in the analysis by the slope-deflection method is the *geometrically symmetric structure*. Structures belong to this class contain a plane, generally called a plane of symmetry, such that the *geometry* of a structure on one side of this plane is an mirror image of that on the other side. The geometry in this sense includes information such as structure configuration (e.g. dimensions and orientations), flexural rigidity  $EI$ , and supports. Structures shown in Figure 12.26 are examples of geometrically symmetric structures; in particular, a plane of symmetry passes through a joint in Figure 12.26(a) and passes through the mid-point of a segment in Figure 12.26(b).

Structures shown in Figure 12.27 seem to be geometrically symmetric; however, with careful investigation, the supports at points A and E of the structure shown in Figure 12.27(a) are different and the flexural rigidity of members AB and CD of the structure shown in Figure 12.27(b) are different. As a result, both of these structures are not geometrically symmetric.

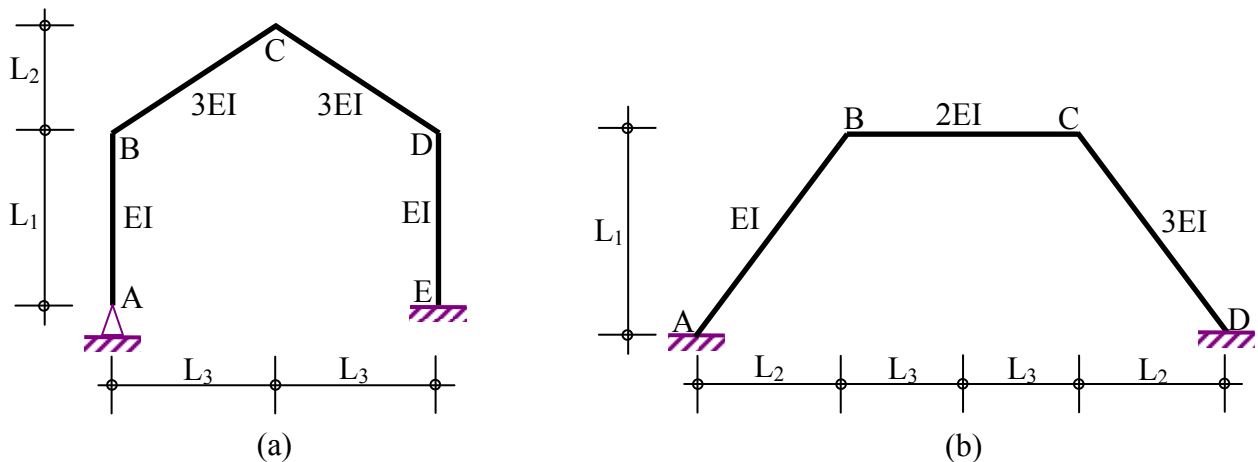


Figure 12.27 Schematic of geometrically non-symmetric structures where (a) supports at points A and E are different and (b) flexural rigidity of members AB and CD are different

### 12.7.2.1 Geometrically symmetric structure under symmetrical loadings

Now, let us focus on a geometrically symmetric structure subjected to a special set of applied loads termed a set of *symmetrical loadings*. An applied load belongs to this set if and only if its mirror image (with respect to the plane of symmetry of a structure) is also contained in this set. By letting A be any point of a geometrically symmetric structure and  $A^*$  be its image point (with respect to its plane of symmetry), following three pairs of applied loads (one acting at point A and the other acting at its image point  $A^*$ ) are obviously symmetrical loadings: (i) a pair of identical forces parallel to the plane of symmetry shown in Figure 12.28(a), (ii) a pair of equal and opposite forces normal to the plane of symmetry shown in Figure 12.28(b), and (iii) a pair of equal and opposite moments shown in Figure 12.28(c).

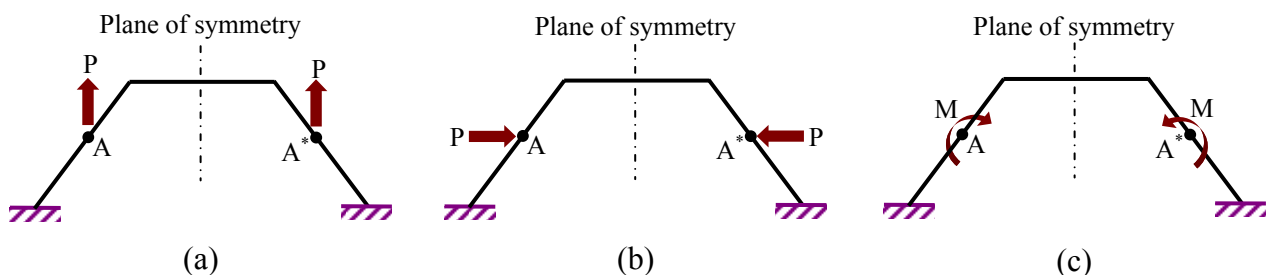


Figure 12.28 Schematics of geometrically symmetric structure under pairs of symmetrical loads

Other types of symmetrical loadings can generally be expressed as linear combinations of above three fundamental pairs. For instance, a pair of inclined symmetrical forces shown in Figure 12.29(a) can be viewed as a combination of a pair of identical forces parallel to the plane of symmetry and a pair of equal and opposite forces normal to the plane of symmetry; the distributed load can be considered as a continuous combination of forces acting to a portion of the structure and it is considered as the symmetrical loading if and only if there exists a continuous combination of their mirror images acting to that structure as shown in Figures 12.29(b) and 12.29(c).

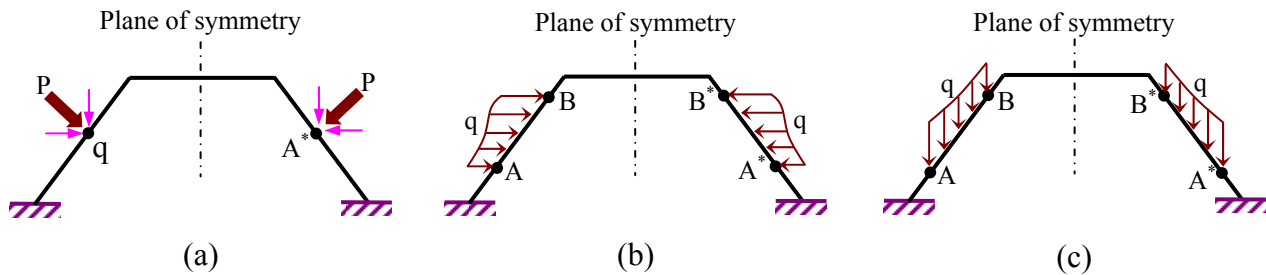


Figure 12.29 Schematics of geometrically symmetric structure under other symmetrical loads

For convenience in further references, a geometrically symmetric structure subjected to a set of symmetrical loadings is termed a *symmetric structure*. This particular structure possesses several features that can later be exploited to reduce the number of primary unknowns and computational effort. For instance, the displacement and rotation at any point of this structure are symmetric in the following sense: (i) a component parallel to the plane of symmetry of the displacement at a point A and its image point A\* are identical as shown in Figure 12.30(a), (ii) a component normal to the plane of symmetry of the displacement at a point A and its image point A\* are equal and opposite as shown in Figure 12.30(b), and (iii) the rotation at a point A and its image point are equal and opposite as shown in Figure 12.30(c). As a result of the symmetry feature of displacement and rotation, side-sway of a discretized structure can only occur in a symmetric mode as shown in Figure 12.31.

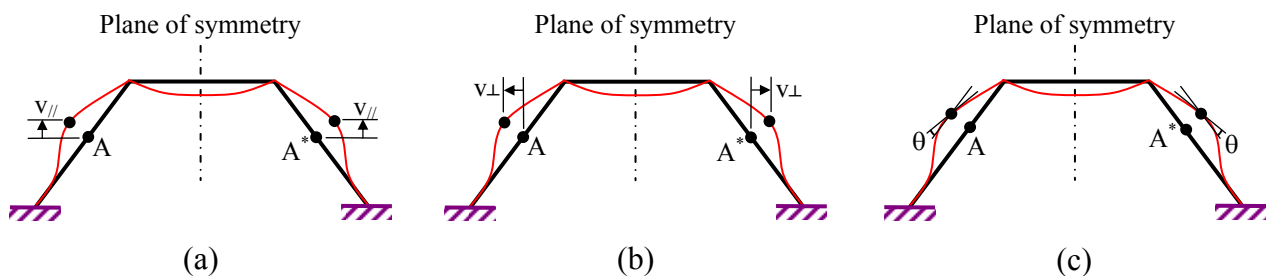


Figure 12.30 Displacement and rotation of symmetric structure

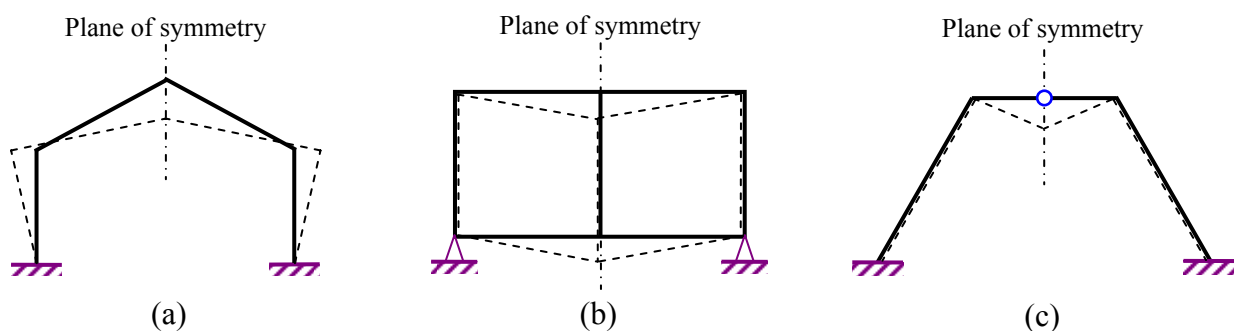


Figure 12.31 Acceptable sway modes for symmetric structures

The internal forces (i.e. the axial force, the bending moment, and the shear force) and all support reactions also possess a symmetric characteristic in the same sense as the external applied forces and moments discussed above. For instance, the axial force, shear force, and bending moment at any point A and at its image point A\* and components of support reactions for a symmetric structure are indicated in Figures 12.32(a), 12.32(b), and 12.32(c), and Figure 12.33, respectively.

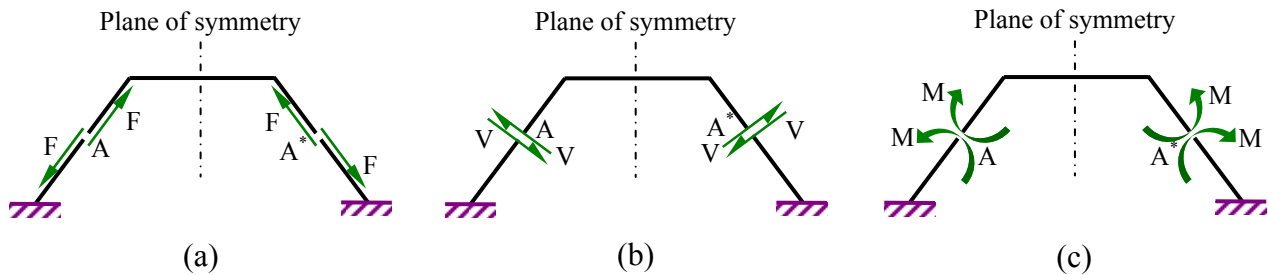


Figure 12.32 Axial force, shear force and bending moment of symmetric structure

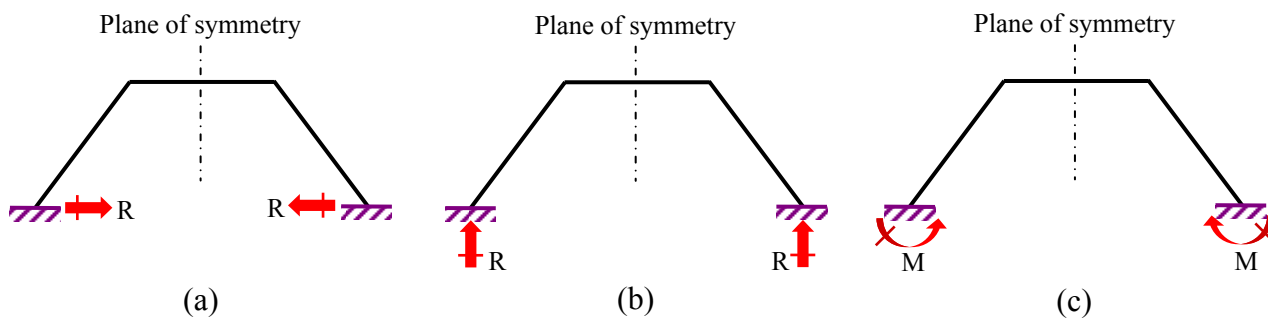


Figure 12.33 Components of support reactions in symmetric structure

By using the symmetric characteristics of geometric properties, loading conditions and responses of a symmetric structure, a large number of primary unknowns and the slope-deflection equations for certain members can be disregarded in the analysis by the slope-deflection method.

For a discretized symmetric structure with its plane of symmetry passing through a node A (see Figure 12.34(a) and Figure 12.35(a)), only a half of the structure (on either side of the plane of symmetry), once the proper treatment of applied loads and movement constraint at node A is considered, can be used in the analysis. From the symmetry, the node A can only move in the direction parallel to the plane of symmetry and if there is no moment release at this node in the original structure, its rotation must vanish (otherwise the rotation is allowed). To mimic such behavior correctly in the reduced structure, the node A must be properly constrained. For instance, the node A of the structure shown in Figure 12.34(a) must be constrained by a guide support in the reduced structure shown in Figure 12.34(b) whereas the node A of the structure shown in Figure 12.35(a) must be constrained by a roller support in the reduced structure shown in Figure 12.35(b). It is noted in addition that only a force in the direction parallel to the plane of symmetry can be applied to the node A (in order to maintain its symmetry) and only a half of this force is applied to the node A in the reduced structure. The reduced structures shown in Figures 12.34(b) and 12.35(b) are now used as the representatives of the original structures shown in Figure 12.34(a) and 12.35(a). It is obvious that the reduced structure contains approximately half of primary unknowns of the original one. Once the reduced structure is analyzed, responses for the other half of the original structure (e.g. end moments, end shear forces, end axial forces, displacements and rotations, support reactions, AFD, SFD, BMD) can readily be obtained from the symmetry conditions.

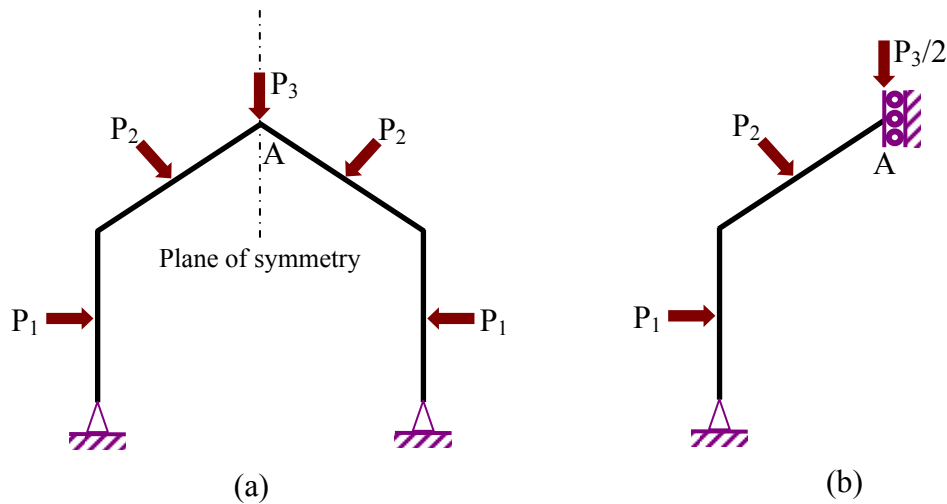


Figure 12.34 (a) Symmetric structure with its plane of symmetry passing through the node A (with no moment release) and (b) equivalent reduced structure

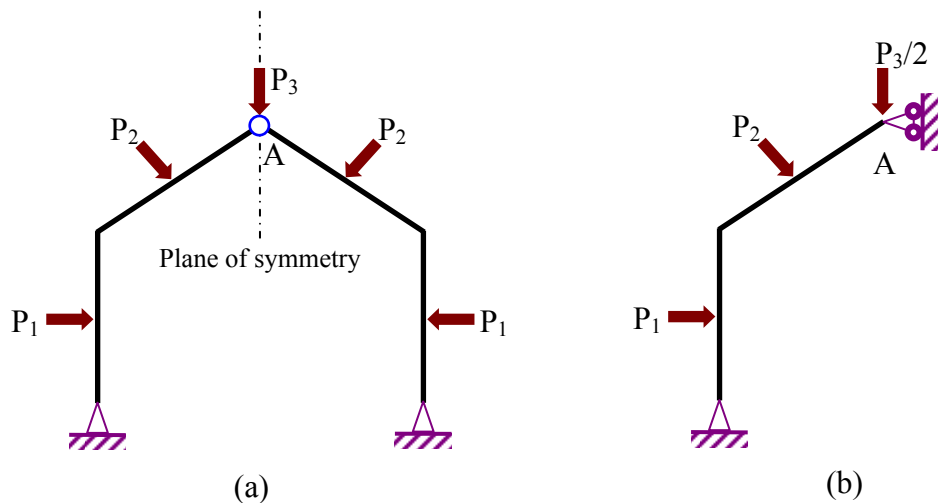


Figure 12.35 (a) Symmetric structure with its plane of symmetry passing through the node A (with moment release) and (b) equivalent reduced structure

For a discretized symmetric structure with its plane of symmetry passing through a center line of a member (see Figure 12.36), the analysis by the slope-deflection method is still performed on the entire structure but the number of primary unknowns and the corresponding equilibrium equations can significantly be reduced from the symmetry conditions. Only primary unknowns and slope-deflection equations associated with a half of the structure (on either side of the plane of symmetry) are sufficient to be treated. For instance, the discretized frame (into members AB, BC and CD) shown in Figure 12.36 originally contains three primary unknowns associated with rotations at nodes B and C and one sway degree of freedom. By exploiting the symmetry, the sway mode cannot occur and the rotations at nodes B and C are not independent but must satisfy the relation  $\theta_B = -\theta_C$ . As a result, the number of primary unknowns can be reduced from three to one; only the rotation  $\theta_B$  can be chosen as a primary unknown. In addition, since the member CD is a mirror image of the member AB, the end rotations, the sway angle and fixed-end moments are directly related to those of the member AB from the symmetry. As a result, the slope-deflection equations of the member CD can be obtained directly from those of the member AB if needed; in general, there is no need to write the slope-deflection equations again for all image members. For the member BC (a member where the plane of symmetry passing through its centerline), it is

obviously symmetric (this implied that the sway angle vanishes and the end rotations are equal and opposite) and the two slope-slope deflection equations simply reduces to (see more details in subsection 12.5.3)

$$M_{BC} = -M_{CB} = \frac{2EI}{L}\theta_B + FEM_{BC} \quad (12.79)$$

It is obvious that the modified slope deflection equation (12.79) contains only one unknown rotation  $\theta_B$ . The modified slope-deflection equation for the symmetric member along with the slope-deflection equations for other members on a half of the structure are sufficient for forming a set of equilibrium equations for determining all (independent) primary unknowns.

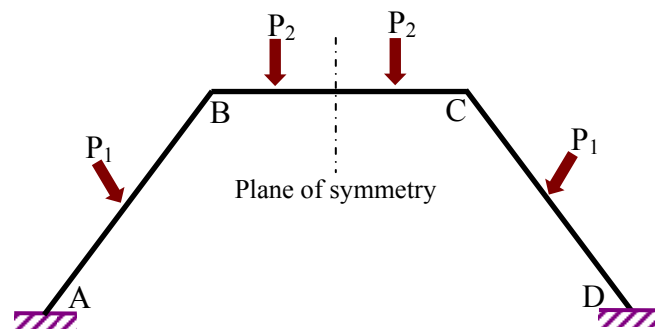


Figure 12.36 Schematic of symmetric structure with the plane of symmetry passing through the centerline of member.

### 12.7.2.2 Geometrically symmetric structure under anti-symmetrical loadings

Next, let consider a geometrically symmetric structure subjected to a special set of applied loads termed a set of *anti-symmetrical loadings*. An applied load belongs to this set if and only if its anti-mirror image (with respect to the plane of symmetry of a structure) is also contained in this set. By letting A be any point of a geometrically symmetric structure and A\* be its image point (with respect to its plane of symmetry), following three pairs of applied loads (one acting at point A and the other acting at its image point A\*) are anti-symmetrical loadings: (i) a pair of equal and opposite forces parallel to the plane of symmetry shown in Figure 12.37(a), (ii) a pair of identical forces normal to the plane of symmetry shown in Figure 12.37(b), and (iii) a pair of identical moments shown in Figure 12.37(c).

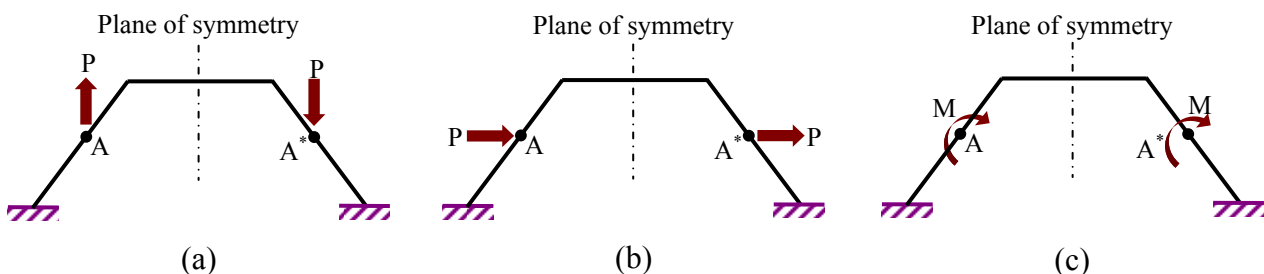


Figure 12.37 Schematics of geometrically symmetric structure under pairs of anti-symmetrical loads

Other types of anti-symmetrical loadings can be expressed as linear combinations of above three fundamental pairs. For instance, a pair of inclined anti-symmetrical forces shown in Figure 12.37(a) can be viewed as a combination of a pair of equal and opposite forces parallel to the plane of

symmetry and a pair of identical forces normal to the plane of symmetry; the distributed load is considered as the anti-symmetrical loading if and only if there exists a continuous combination of their anti-mirror images acting to that structure as shown in Figures 12.38(b) and 12.38(c).

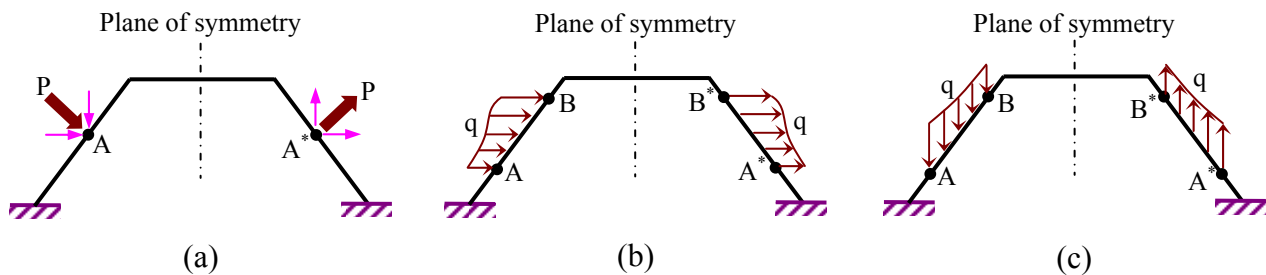


Figure 12.38 Schematics of geometrically symmetric structure under other anti-symmetrical loads

For convenience in further references, a geometrically symmetric structure subjected to a set of anti-symmetrical loadings is termed an *anti-symmetric structure*. This type of structures possesses several characteristics that can be exploited to reduce the number of primary unknowns and computational effort. For instance, the displacement and rotation at any point of this structure are anti-symmetric in the following sense: (i) a component parallel to the plane of symmetry of the displacement at a point A and its image point A\* are equal and opposite as shown in Figure 12.39(a), (ii) a component normal to the plane of symmetry of the displacement at a point A and its image point A\* are identical as shown in Figure 12.39(b), and (iii) the rotation at a point A and its image point are identical as shown in Figure 12.39(c). As a result of the anti-symmetry conditions for the displacement and rotation, side-sway of a discretized structure can only occur in an anti-symmetric mode as indicated in Figure 12.40.

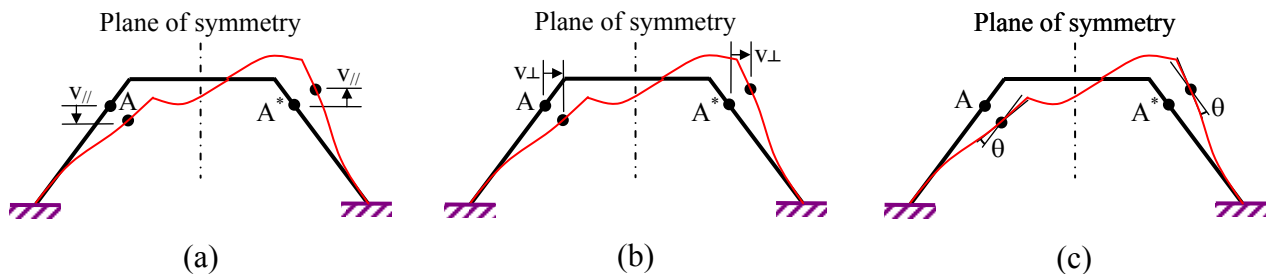


Figure 12.39 Displacement and rotation of anti-symmetric structure

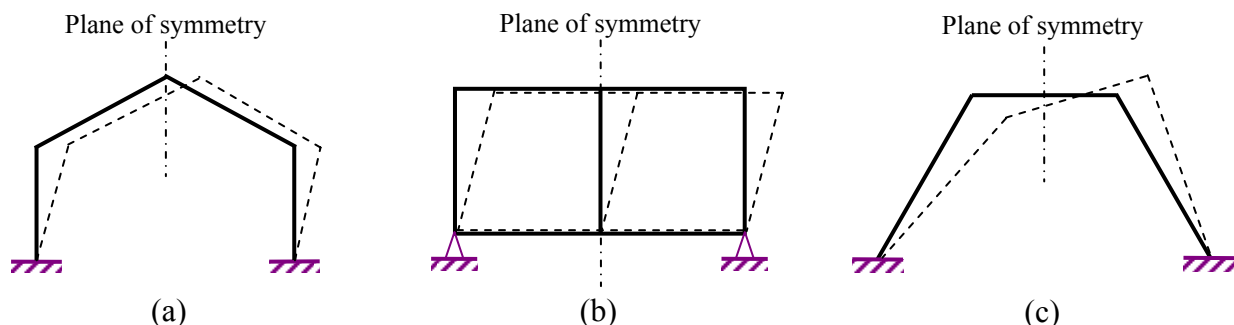


Figure 12.40 Acceptable sway modes for anti-symmetric structures

The internal forces (i.e. the axial force, the bending moment, and the shear force) and all support reactions also possess a anti-symmetric characteristic in the same sense as external applied forces

and moments discussed above. For instance, the axial force, shear force, and bending moment at any point A and at its image point A\* and components of support reactions for an anti-symmetric structure are indicated in Figures 12.41(a), 12.41(b), and 12.41(c), and Figure 12.42, respectively.

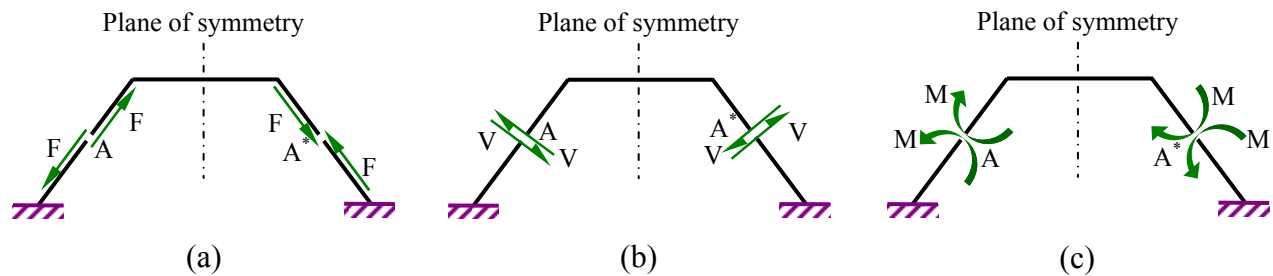


Figure 12.41 Axial force, shear force and bending moment of anti-symmetric structure

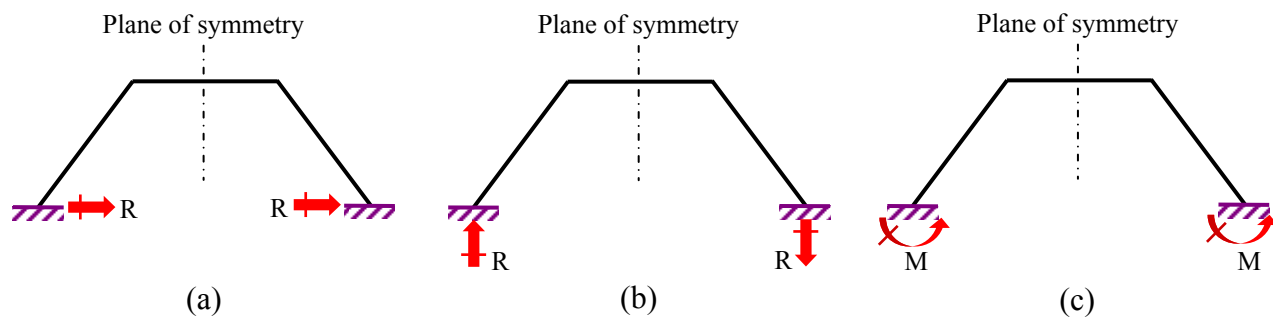


Figure 12.42 Components of support reactions in symmetric structure

By using the anti-symmetric characteristics of geometric properties, loading conditions and responses of an anti-symmetric structure, a large number of primary unknowns and the slope-deflection equations for certain members can also be disregarded in the analysis by the slope-deflection method.

For a discretized anti-symmetric structure with its plane of symmetry passing through a node A (see Figure 12.43(a)), only a half of the structure (on either side of the plane of symmetry), once the proper treatment of applied loads and movement constraint at node A is considered, can be used in the analysis. From the anti-symmetry behavior, the node A can only move in the direction normal to the plane of symmetry and the bending moment at point A vanishes. To mimic such behavior correctly in the reduced structure, the node A must be properly constrained. For instance, the node A of the structure shown in Figure 12.43(a) must be constrained by a roller support in the reduced structure shown in Figure 12.43(b). It should be noted in addition that only a force in the direction normal to the plane of symmetry and a moment can be applied to the node A (a force parallel to the plane of symmetry can be applied to the node A since it destroys the anti-symmetry feature) and only a half of this force and moment is applied to the node A in the reduced structure. The reduced structure shown in Figures 12.43(b) is now used as the representative of the original structures shown in Figure 12.43(a). It is obvious that the reduced structure contains approximately half of primary unknowns of the original one. Once the reduced structure is analyzed, responses for the other half of the original structure (e.g. end moments, end shear forces, end axial forces, displacements and rotations, support reactions, AFD, SFD, BMD) can readily be obtained from the anti-symmetry conditions.

For a discretized symmetric structure with its plane of symmetry passing through a center line of a member (see Figure 12.44), the analysis by the slope-deflection method is still performed



on the entire structure similar to the case of symmetric structures but again the number of primary unknowns and the corresponding equilibrium equations can significantly be reduced from the anti-symmetry conditions. Only primary unknowns and slope-deflection equations associated with a half of the structure (on either side of the plane of symmetry) are sufficient to be considered. For instance, the discretized frame (into members AB, BC and CD) shown in Figure 12.44 originally contains three primary unknowns associated with rotations at nodes B and C and one sway degree of freedom. By exploiting the anti-symmetry conditions, the rotations at nodes B and C are not independent but must satisfy the relation  $\theta_B = \theta_C$ . As a result, the number of primary unknowns can be reduced from three to two; only the rotation  $\theta_B$  and the sway degree of freedom can be chosen as primary unknowns. In addition, since the member CD is a mirror image of the member AB, the end rotations, the sway angle and fixed-end moments are directly related to those of the member AB from the anti-symmetry properties. As a result, the slope-deflection equations of the member CD can be obtained directly from those of the member AB if needed; in general, there is no need to write the slope-deflection equations again for all image members. For the member BC (a member where the plane of symmetry passing through its centerline), it is obviously anti-symmetric (the end rotations are identical) and the two slope-slope deflection equations simply reduces to (see more details in subsection 12.5.4)

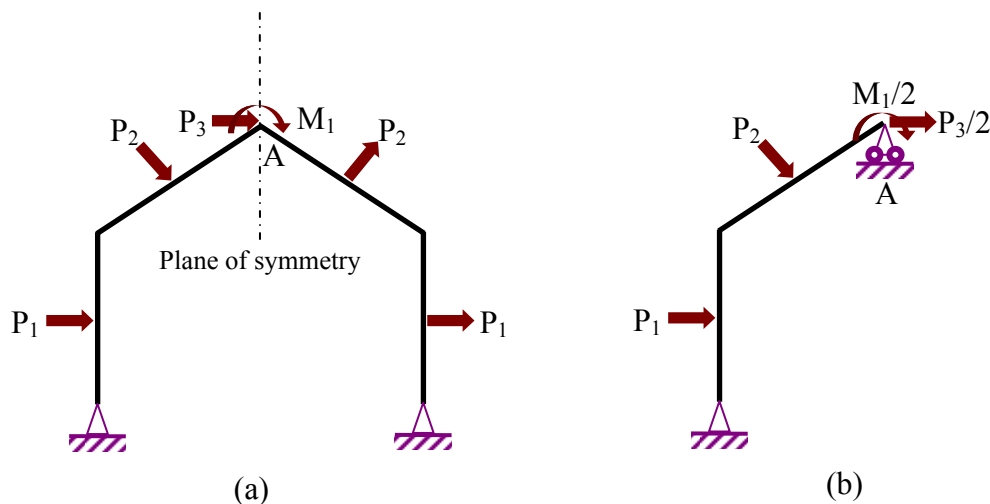


Figure 12.43 (a) Anti-symmetric structure with its plane of symmetry passing through the node A and (b) equivalent reduced structure

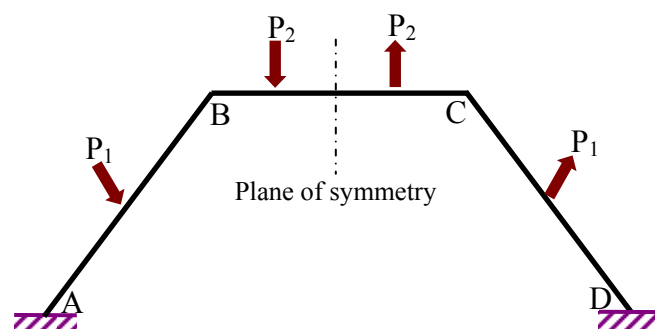


Figure 12.44 Anti-schematic of symmetric structure with the plane of symmetry passing through the centerline of member.

$$M_{BC} = M_{CB} = \frac{6EI}{L} \theta_B - \frac{6EI}{L} \psi_{BC} + FEM_{BC} \quad (12.80)$$



It is obvious that the modified slope deflection equation (12.80) contains only one unknown rotation  $\theta_B$  and the sway angle  $\psi_{BC}$ . Again, the modified slope-deflection equation for the anti-symmetric member along with the slope-deflection equations for other members on a half of the structure are sufficient for forming a set of equilibrium equations for determining all (independent) primary unknowns.

### 12.7.2.3 Geometrically symmetric structure under general loadings

Finally, we consider a geometrically symmetric structure subjected to a set of *general loadings*. It can readily be verified that this structure can always be decomposed into two structures, one subjected to a set of symmetrical loadings and the other subjected to a set of anti-symmetrical loadings. Such decomposition can be achieved by using following fundamental results: (i) a concentrated force parallel to the plane of symmetry acting at a point A can be decomposed into a pair of symmetrical forces and a pair of anti-symmetrical forces, one acting at the point A and the other acting at its image point  $A^*$  as shown in Figure 12.45; (ii) a concentrated force normal to the plane of symmetry acting at a point A can be decomposed into a pair of symmetrical forces and a pair of anti-symmetrical forces, one acting at the point A and the other acting at its image point  $A^*$  as shown in Figure 12.46; and (iii) a concentrated moment acting at a point A can be decomposed into a pair of symmetrical moments and a pair of anti-symmetrical moments, one acting at the point A and the other acting at its image point  $A^*$  as shown in Figure 12.47.

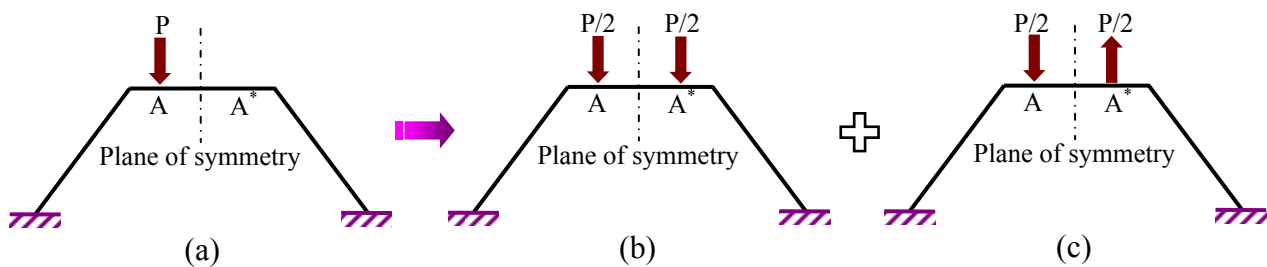


Figure 12.45 Decomposition of concentrated force parallel to plane of symmetry into (b) a pair of symmetrical forces and (c) a pair of anti-symmetrical forces

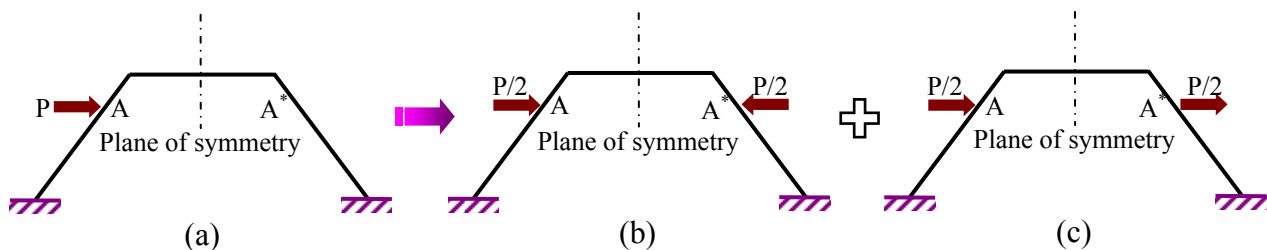


Figure 12.46 Decomposition of concentrated force normal to plane of symmetry into (b) a pair of symmetrical forces and (c) a pair of anti-symmetrical forces

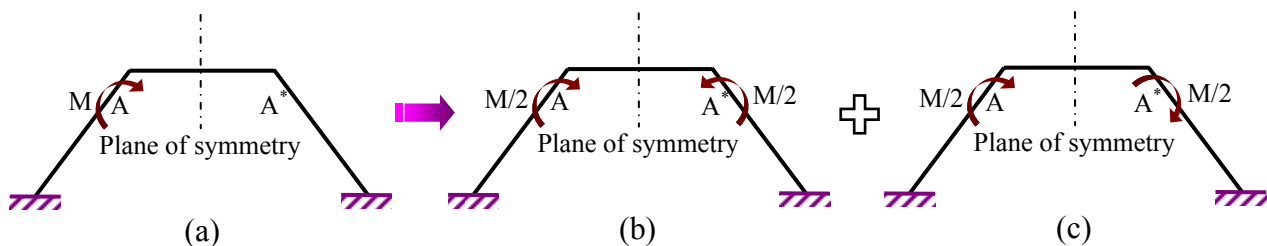


Figure 12.47 Decomposition of concentrated moment into (b) a pair of symmetrical moments and (c) a pair of anti-symmetrical moments

Other types of loadings can also be decomposed into symmetric and anti-symmetric loadings by employing above three results as a basis. For instance, by using the results (i) and (ii), a concentrated force inclined to the plane of symmetry can be decomposed into a pair of inclined symmetrical forces and a pair of inclined anti-symmetrical forces as shown in Figure 12.48. Also, by treating the distributed force as a continuous distribution of concentrated forces, it can be decomposed into a pair of symmetrical distributed forces and a pair of anti-symmetrical distributed forces as shown in Figures 12.49 and 12.50.

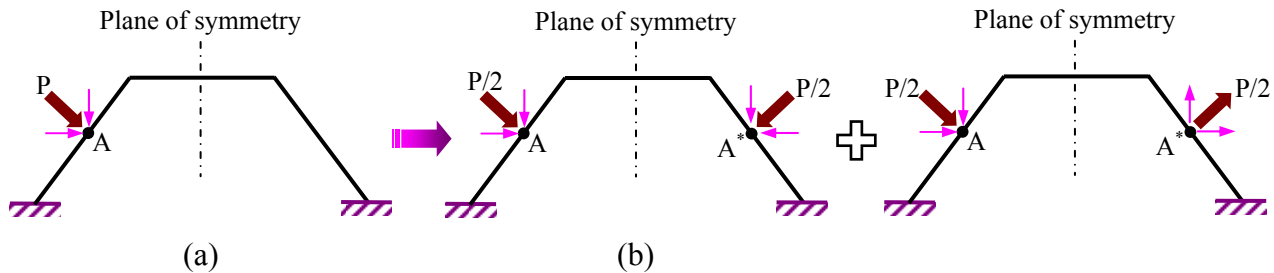


Figure 12.48 Decomposition of concentrated force inclined to the plane of symmetry into (b) a pair of symmetrical distributed forces and (c) a pair of anti-symmetrical distributed forces

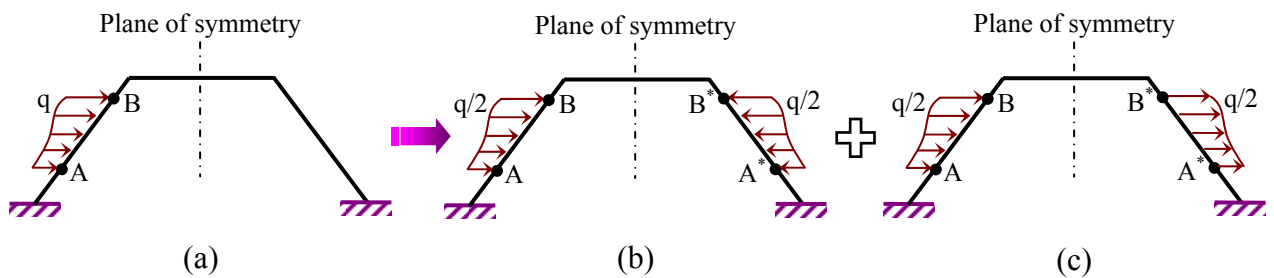


Figure 12.49 Decomposition of distributed force normal to the plane of symmetry into (b) a pair of symmetrical distributed forces and (c) a pair of anti-symmetrical distributed forces

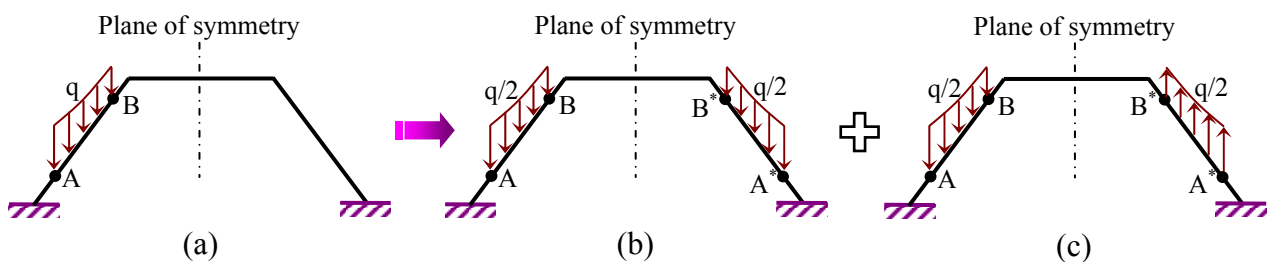


Figure 12.50 Decomposition of distributed force parallel to the plane of symmetry into (b) a pair of symmetrical distributed forces and (c) a pair of anti-symmetrical distributed forces

By exploiting above results, geometrically symmetric frames subjected to general loadings as shown in Figures 12.51(a) and 12.52(a) can be decomposed into symmetric structures shown in Figures 12.51(b) and 12.52(b) and anti-symmetric structures shown in Figures 12.51(c) and 12.52(c). Analysis of the symmetric and anti-symmetric structures by the slope-deflection method follows procedures and guidelines described in subsections 12.7.2.1 and 12.7.2.2. From the principle of superposition for linear structures, responses of the original structure can readily be obtained by combining responses of the corresponding symmetric and anti-symmetric structures.

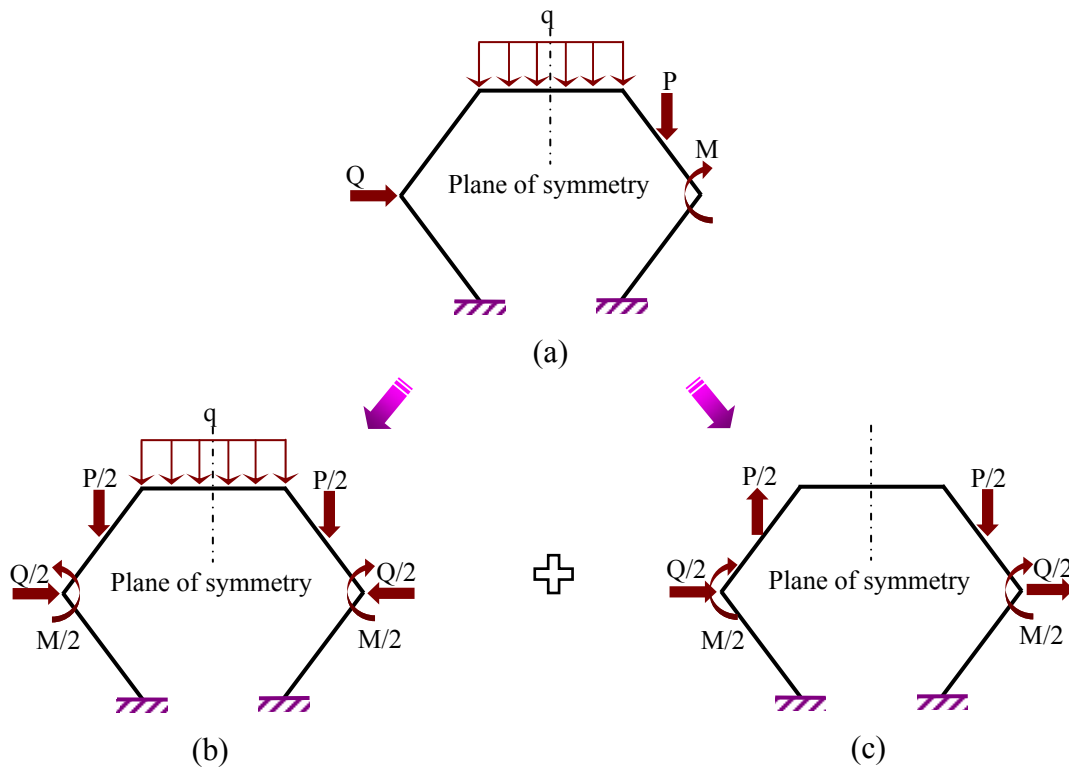


Figure 12.51 Decomposition of geometrically symmetric structure under general loadings into (b) symmetric structure and (c) anti-symmetric structure

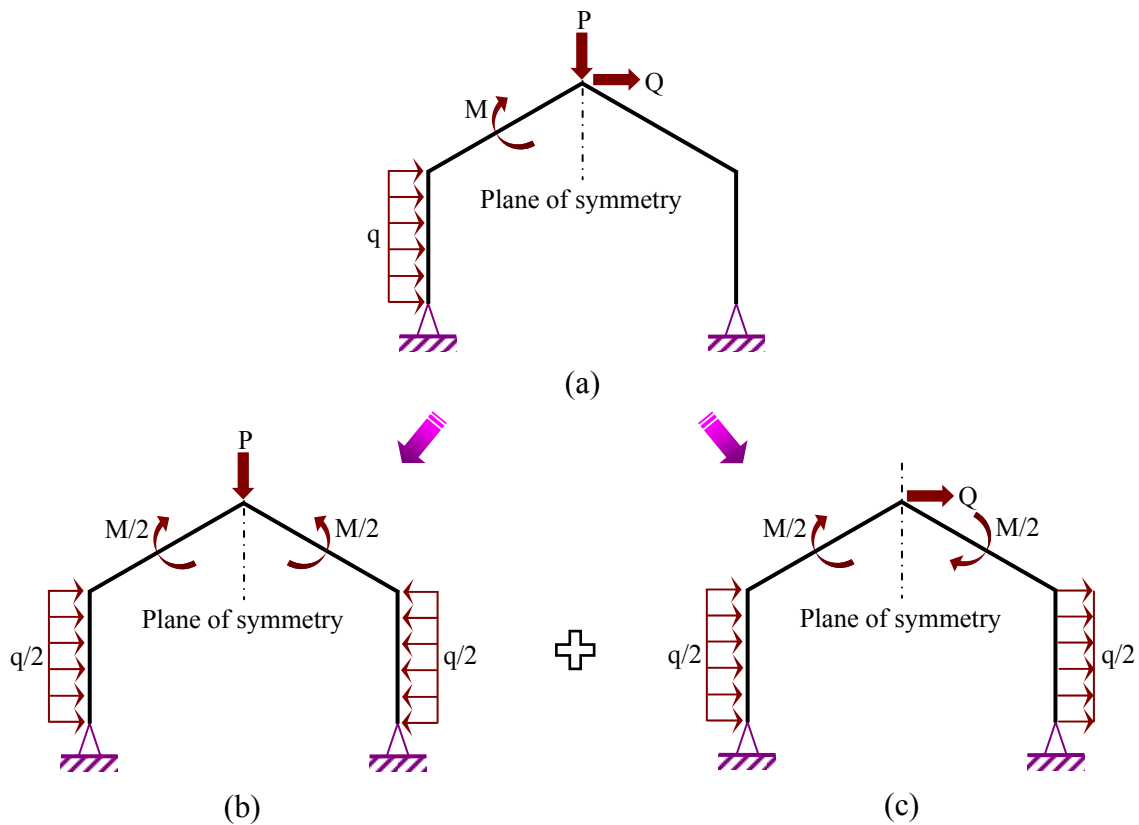
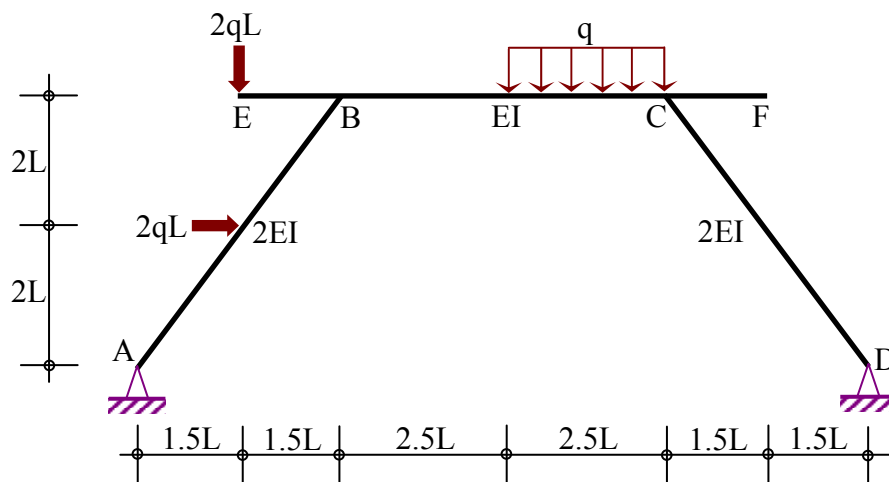
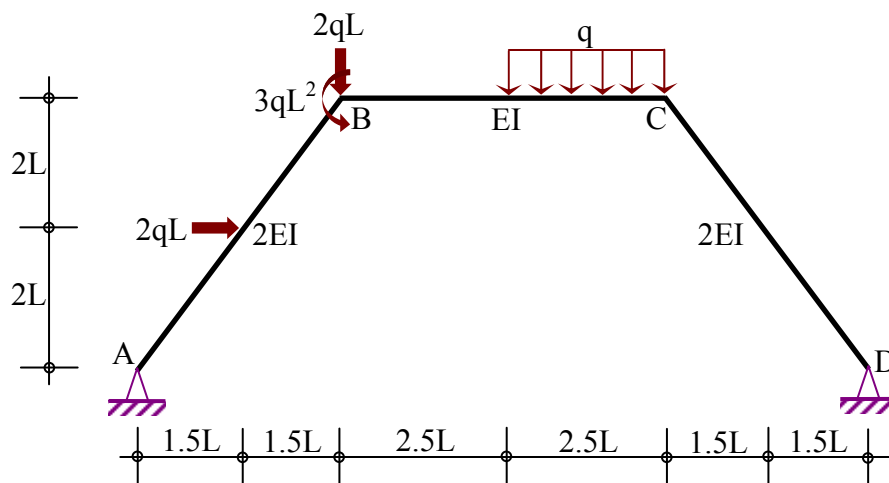


Figure 12.52 Decomposition of geometrically symmetric structure under general loadings into (b) symmetric structure and (c) anti-symmetric structure

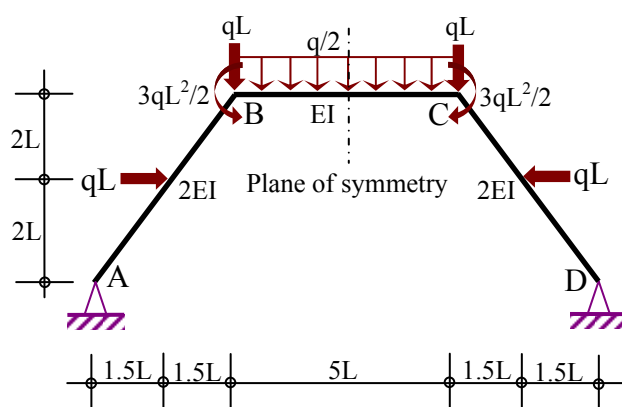
**Example 12.15** Use the slope-deflection method along with symmetric and anti-symmetric features to analyze a frame subjected to external loads shown below. The flexural rigidity is clearly indicated in the figure below.



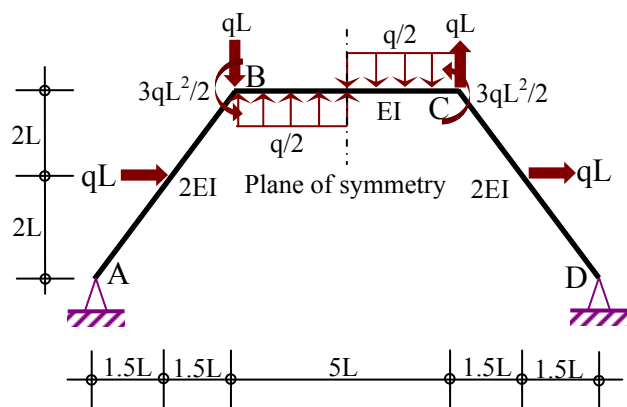
**Solution** Since the overhanging segments EB and CF are statically determinate, they can be eliminated from the original structure and then replaced by equivalent force and moment as shown below. It is obvious that the resulting structure is geometrically symmetric and can therefore be decomposed into symmetric and anti-symmetric structures as indicated below.



Reduced Structure



Symmetric Structure



Anti-symmetric Structure

**Symmetric Structure:** The structure is discretized into three members AB, BC and CD with four nodes A, B, C, and D. Based on this discretization, the number of rotational degrees of freedom is equal to  $N_r = 4 + 0 - 0 = 4$  (i.e.  $\theta_A$ ,  $\theta_B$ ,  $\theta_C$  and  $\theta_D$ ) and the number of sway degrees of freedom is equal to  $N_s = 2(4) - 4 - 3 = 1$ ; thus, the number of primary unknowns is equal to  $4 + 1 = 5$ . Due to the symmetry, the side-sway of this frame cannot occur for this set of symmetrical loadings (i.e. the sway degree of freedom is known to be zero a priori) and, in addition, the rotations at four nodes must satisfy  $\theta_A = -\theta_D$  and  $\theta_B = -\theta_C$ . Therefore, the number of primary unknowns can be reduced from 5 to 2; only the rotations  $\theta_A$  and  $\theta_B$  are independent primary unknowns. By noting that the bending moment at node A vanishes, the modified slope-deflection (12.74) or (12.76) can be applied to the member AB and the rotation  $\theta_A$  can further be eliminated from a set of primary unknowns. In summary, by using the symmetry of the structure and loadings along with the prescribed bending moment at node A, the number of primary unknowns is significantly reduced from 5 to 1.

The modified slope-deflection equations for member AB (a member with prescribed bending moment at one end) and member BC (a symmetric member) are given below.

**Member AB** For this member,  $M_{A0} = 0$ ,  $\psi_{AB} = 0$ ,  $FEM_{AB} = (qL)(4L)/8 = qL^2/2$ ,  $FEM_{BA} = -(qL)(4L)/8 = -qL^2/2$  and the modified fixed-end moment at the end B is given by

$$\overline{FEM}_{BA} = FEM_{BA} - \frac{1}{2}FEM_{AB} + \frac{1}{2}M_{A0} = -\frac{qL^2}{2} - \frac{1}{2}\left(\frac{qL^2}{2}\right) + \frac{1}{2}(0) = -\frac{3qL^2}{4}$$

The modified slope-deflection equations become

$$M_{BA} = \frac{3(2EI)}{5L}\theta_B - \frac{3(2EI)}{5L}(0) - \frac{3qL^2}{4} = \frac{6EI\theta_B}{5L} - \frac{3qL^2}{4} \quad (e12.15.1)$$

The rotation at the end A is given by

$$\theta_A = -\frac{1}{2}\theta_B + \frac{3}{2}(0) - \frac{5L}{4(2EI)}\left(\frac{qL^2}{2} - 0\right) = -\frac{1}{2}\theta_B - \frac{5qL^3}{16EI} \quad (e12.15.2)$$

**Member BC** For this member,  $FEM_{BC} = (q/2)(5L)^2/12 = 25qL^2/24$  and the modified slope-deflection equations become

$$M_{BC} = \frac{2(EI)}{5L}\theta_B + \frac{25qL^2}{24} = \frac{2EI\theta_B}{5L} + \frac{25qL^2}{24} \quad (e12.15.3)$$

A single equilibrium equation needed for solving the primary unknown  $\theta_B$  can readily be obtained from moment equilibrium at node B as follows:

$$\begin{aligned} [\Sigma M_{\text{node B}} = 0] &\Rightarrow M_{BA} + M_{BC} - 3qL^2/2 = 0 \\ &\Rightarrow \left(\frac{6EI\theta_B}{5L} - \frac{3qL^2}{4}\right) + \left(\frac{2EI\theta_B}{5L} + \frac{25qL^2}{24}\right) - \frac{3qL^2}{2} = 0 \\ &\Rightarrow \frac{8EI\theta_B}{5L} = \frac{29qL^2}{24} \end{aligned} \quad (e12.15.4)$$

By solving equation (e12.15.4), it yields

$$\frac{EI\theta_B}{L} = \frac{145qL^2}{192} \quad (e12.15.5)$$

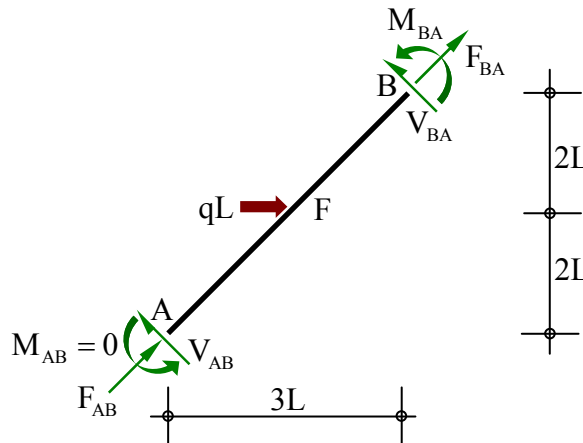
The rotation at the end A is obtained from (e12.15.2) as

$$\theta_A = -\frac{1}{2}\theta_B - \frac{5qL^3}{16EI} = -\frac{1}{2}\left(\frac{145qL^3}{192EI}\right) - \frac{5qL^3}{16EI} = -\frac{265qL^3}{384EI} \quad (\text{e12.15.6})$$

The end moments, end shear forces, end axial forces, and support reactions for the left half of the structure can be obtained as follows:

Member AB

Free body diagram:



End moments:  $M_{AB} = 0$  and

$$(\text{e12.15.1}) \Rightarrow M_{BA} = \frac{6EI\theta_B}{5L} - \frac{3qL^2}{4} = \frac{6}{5}\left(\frac{145qL^2}{192}\right) - \frac{3qL^2}{4} = \frac{5qL^2}{32}$$

End shear forces:

$$[\Sigma M_{\text{end B}} = 0] \Rightarrow V_{AB}(5L) - M_{AB} - M_{BA} - qL(2L) = 0 \Rightarrow V_{AB} = 69qL/160$$

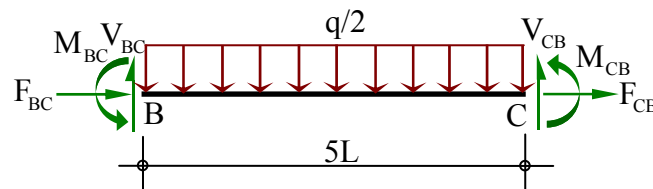
$$[\Sigma F_{y, \text{ member AB}} = 0] \Rightarrow V_{AB} + V_{BA} - qL(4/5) = 0 \Rightarrow V_{BA} = 59qL/160$$

End axial forces:

$$[\Sigma F_{x, \text{ member AB}} = 0] \Rightarrow F_{AB} + F_{BA} + qL(3/5) = 0 \quad (\text{e12.15.7})$$

Member BC

Free body diagram:

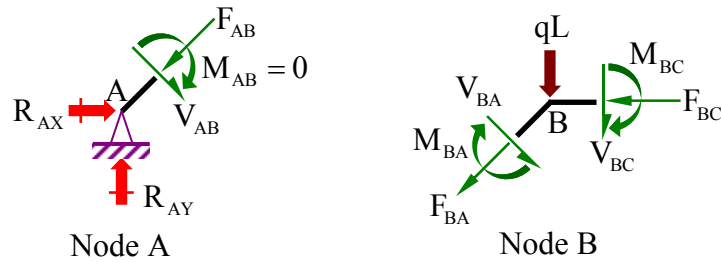


End moments:

$$(\text{e12.15.3}) \Rightarrow M_{BC} = \frac{2EI\theta_B}{5L} + \frac{25qL^2}{24} = \frac{2}{5}\left(\frac{145qL^2}{192}\right) + \frac{25qL^2}{24} = \frac{43qL^2}{32}$$

End shear forces:

$$[\Sigma F_{y, \text{ member BC}} = 0] \Rightarrow V_{BC} + V_{CB} - (q/2)(5L) = 2V_{BC} - (q/2)(5L) = 0 \Rightarrow V_{BC} = 5qL/4$$



$$\begin{aligned} \text{Node B: } [\Sigma F_{y, \text{node B}} = 0] &\Rightarrow -F_{BA}(4/5) - V_{BA}(3/5) - V_{BC} - qL = 0 \Rightarrow F_{BA} = -1977qL/640 ; \\ [\Sigma F_{x, \text{node B}} = 0] &\Rightarrow -F_{BA}(3/5) + V_{BA}(4/5) - F_{BC} = 0 \Rightarrow F_{BC} = 275qL/128 \end{aligned}$$

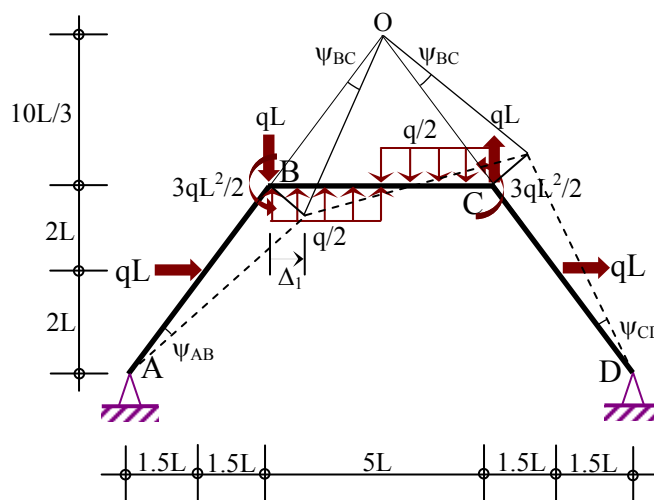
$$(e12.15.7) \Rightarrow F_{AB} = -F_{BA} - qL(3/5) = 1593qL/640$$

$$\begin{aligned} \text{Node A: } R_{AX} &= F_{AB}(3/5) - V_{AB}(4/5) = 147qL/128 \text{ Rightward} ; \\ R_{AY} &= F_{AB}(4/5) + V_{AB}(3/5) = 9qL/4 \text{ Upward} \end{aligned}$$

By exploiting the symmetry, end moments, end shear forces, end axial forces and support reactions for the right half of the structure can readily be obtained as follows:

$$\begin{aligned} M_{DC} &= -M_{AB} = 0 ; M_{CD} = -M_{BA} = -5qL^2/32 ; M_{CB} = -M_{BC} = -43qL^2/32 \\ V_{DC} &= V_{AB} = 69qL/160 ; V_{CD} = V_{BA} = 59qL/160 ; V_{CB} = V_{BC} = 5qL/4 \\ F_{DC} &= -F_{AB} = -1593qL/640 ; F_{CD} = -F_{BA} = 1977qL/640 ; F_{CB} = -F_{BC} = -275qL/128 \\ R_{DX} &= -R_{AX} = -147qL/128 ; R_{DY} = R_{AY} = 9qL/4 \end{aligned}$$

**Anti-symmetric Structure:** Again, the structure is discretized into three members AB, BC and CD with four nodes A, B, C, and D and the number of primary unknowns is equal to 5. Due to the anti-symmetry features, the side-sway of this frame is allowed and, in addition, rotations at four nodes must satisfy  $\theta_A = \theta_D$  and  $\theta_B = \theta_C$ . As a result, the number of primary unknowns can be reduced from 5 to 3; only the rotations  $\theta_A$  and  $\theta_B$  and one sway degree of freedom are independent primary unknowns. By using a condition associated with the prescribed bending moment at node A, the number of primary unknowns can further be reduced from 3 to 2; in particular, the rotation  $\theta_A$  can be eliminated from a set of primary unknowns. The sway pattern associated with the sway degree of freedom (the horizontal displacement at node B, denoted by  $\Delta_1$ ) can be sketched as shown below.



Sway pattern

By using the ICR concept, the sway angle of members on the left half of the structure can be expressed in terms of the sway degree of freedom  $\Delta_1$  as follows. Since A and D are fixed points, they are the ICR of members AB and BC, respectively. Using this information, it is known that the movement of point B is perpendicular to the member AB and the movement of point C is perpendicular to the member CD. As a result, the ICR of a member BC is the intersection of the line AB and the line CD, denoted by a point O. By using the relation (12.68) along with the known ICR and known horizontal displacement at point B, the sway angle of member AB and BC are obtained as  $-\Delta_1/4L$  and  $3\Delta_1/10L$ , respectively. The modified slope-deflection equations for member AB (a member with prescribed bending moment at one end) and member BC (an anti-symmetric member) are given below.

Member AB For this member,  $M_{A0} = 0$ ,  $\psi_{AB} = -\Delta_1/4L$ ,  $FEM_{AB} = (qL)(4L)/8 = qL^2/2$ ,  $FEM_{BA} = -(qL)(4L)/8 = -qL^2/2$  and the modified fixed-end moment at the end B is given by

$$\overline{FEM}_{BA} = FEM_{BA} - \frac{1}{2}FEM_{AB} + \frac{1}{2}M_{A0} = -\frac{qL^2}{2} - \frac{1}{2}\left(\frac{qL^2}{2}\right) + \frac{1}{2}(0) = -\frac{3qL^2}{4}$$

The modified slope-deflection equations become

$$M_{BA} = \frac{3(2EI)}{5L}\theta_B - \frac{3(2EI)}{5L}\left(-\frac{\Delta_1}{4L}\right) - \frac{3qL^2}{4} = \frac{6EI\theta_B}{5L} + \frac{3EI\Delta_1}{10L^2} - \frac{3qL^2}{4} \quad (e12.15.8)$$

The rotation at the end A is given by

$$\theta_A = -\frac{1}{2}\theta_B + \frac{3}{2}\left(-\frac{\Delta_1}{4L}\right) - \frac{5L}{4(2EI)}\left(\frac{qL^2}{2} - 0\right) = -\frac{1}{2}\theta_B - \frac{3\Delta_1}{8L} - \frac{5qL^3}{16EI} \quad (e12.15.9)$$

Member BC For this member,  $\psi_{BC} = 3\Delta_1/10L$  and

$$FEM_{BC} = -\frac{(q/2)(5L)^2}{12}\left[6\left(\frac{1}{2}\right)^2 - 8\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^4\right] + \frac{(q/2)(5L)^2}{12}\left[4\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^4\right] = -\frac{25qL^2}{64}$$

The modified slope-deflection equations become

$$M_{BC} = \frac{6EI}{5L}\theta_B - \frac{6EI}{5L}\left(\frac{3\Delta_1}{10L}\right) - \frac{25qL^2}{64} = \frac{6EI\theta_B}{5L} - \frac{9EI\Delta_1}{25L^2} - \frac{25qL^2}{64} \quad (e12.15.10)$$

The first equation associated with moment equilibrium at node B is obtained as follows:

$$\begin{aligned} [\Sigma M_{\text{node B}} = 0] &\Rightarrow M_{BA} + M_{BC} - 3qL^2/2 = 0 \\ &\Rightarrow \left(\frac{6EI\theta_B}{5L} + \frac{3EI\Delta_1}{10L^2} - \frac{3qL^2}{4}\right) + \left(\frac{6EI\theta_B}{5L} - \frac{9EI\Delta_1}{25L^2} - \frac{25qL^2}{64}\right) - \frac{3qL^2}{2} = 0 \\ &\Rightarrow \frac{12EI\theta_B}{5L} - \frac{3EI\Delta_1}{50L^2} = \frac{169qL^2}{64} \end{aligned} \quad (e12.15.11)$$

The second equilibrium equation associated with the sway degree of freedom is obtained by using the principle of virtual work along with the sway pattern being chosen as the virtual displacement:

$$\delta W_{\text{ext}} = 2\{(qL)|\psi_{AB}|(2L) + (qL)|\psi_{AB}|(3L) - (q/2)(5L/2)|\psi_{BC}|(5L/4)\} = 25qL\Delta_1/16$$

$$\delta W_{\text{int}} = -2(M_{AB} + M_{BA})\psi_{AB} - (2M_{BC})\psi_{BC} = M_{BA}\Delta_1/2L - 3M_{BC}\Delta_1/5L$$



$$\begin{aligned}
 \delta W_{\text{ext}} = \delta W_{\text{int}} &\Rightarrow 25qL\Delta_1 / 16 = M_{BA}\Delta_1 / 2L - 3M_{BC}\Delta_1 / 5L \\
 &\Rightarrow M_{BA} / 2 - 3M_{BC} / 5 = 25qL^2 / 16 \\
 &\Rightarrow -\frac{3EI\theta_B}{25L} + \frac{183EI\Delta_1}{500L^2} = \frac{109qL^2}{64}
 \end{aligned} \tag{e12.15.12}$$

By solving a system of linear equations (e12.15.11) and (e12.15.12), it yields

$$\frac{EI\theta_B}{L} = 1.227qL^2$$

$$\frac{EI\Delta_1}{L^2} = 5.056qL^2$$

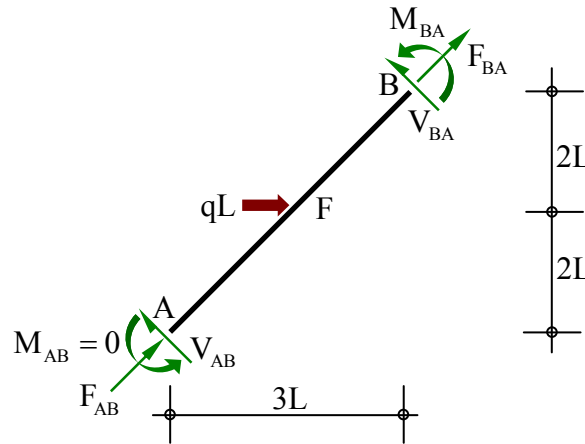
The rotation at the end A is obtained from (e12.15.9) as

$$\theta_A = -\frac{1}{2}\theta_B - \frac{3\Delta_1}{8L} - \frac{5qL^3}{16EI} = -\frac{1}{2}\left(\frac{1.227qL^3}{EI}\right) - \frac{3}{8L}\left(\frac{5.056qL^4}{EI}\right) - \frac{5qL^3}{16EI} = -\frac{2.822qL^3}{EI}$$

The end moments, end shear forces, end axial forces, and support reactions for the left half of the structure can be obtained as follows:

#### Member AB

Free body diagram:



End moments:  $M_{AB} = 0$  and

$$(e12.15.8) \Rightarrow M_{BA} = \frac{6EI\theta_B}{5L} + \frac{3EI\Delta_1}{10L^2} - \frac{3qL^2}{4} = \frac{6}{5}(1.227qL^2) + \frac{3}{10}(5.056qL^2) - \frac{3qL^2}{4} = 2.239qL^2$$

End shear forces:

$$[\Sigma M_{\text{end B}} = 0] \Rightarrow V_{AB}(5L) - M_{AB} - M_{BA} - qL(2L) = 0 \Rightarrow V_{AB} = 0.848qL$$

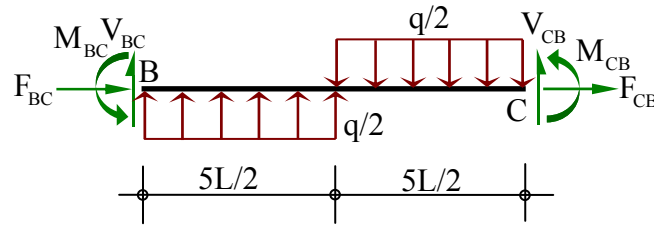
$$[\Sigma F_{y, \text{ member AB}} = 0] \Rightarrow V_{AB} + V_{BA} - qL(4/5) = 0 \Rightarrow V_{BA} = -0.048qL$$

End axial forces:

$$[\Sigma F_{x, \text{ member AB}} = 0] \Rightarrow F_{AB} + F_{BA} + qL(3/5) = 0 \tag{e12.15.13}$$

### Member BC

Free body diagram:

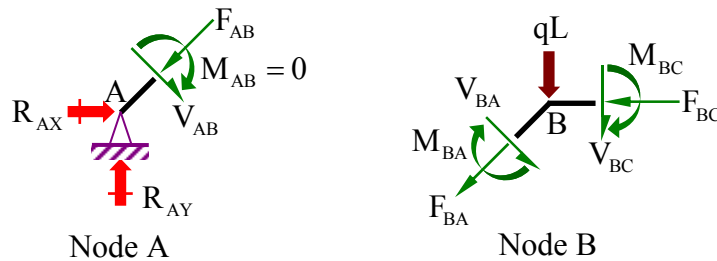


End moments:

$$(e12.15.10) \Rightarrow M_{BC} = \frac{6EI\theta_B}{5L} - \frac{9EI\Delta_1}{25L^2} - \frac{25qL^2}{64} = \frac{6}{5}(1.227qL^2) - \frac{9}{25}(5.056qL^2) - \frac{25qL^2}{64} = -0.738qL^2$$

End shear forces:

$$[\Sigma M_{\text{end C}} = 0] \Rightarrow V_{BC}(5L) - M_{BC} - M_{CB} + (5qL/4)(15L/4) - (5qL/4)(5L/4) = 0 \Rightarrow V_{BC} = -0.920qL$$



$$\text{Node B: } [\Sigma F_{y, \text{node B}} = 0] \Rightarrow -F_{BA}(4/5) - V_{BA}(3/5) - V_{BC} - qL = 0 \Rightarrow F_{BA} = -0.064qL ;$$

$$[\Sigma F_{x, \text{node B}} = 0] \Rightarrow -F_{BA}(3/5) + V_{BA}(4/5) - F_{BC} = 0 \Rightarrow F_{BC} = 0$$

$$(e12.15.13) \Rightarrow F_{AB} = -F_{BA} - qL(3/5) = -0.536qL$$

$$\text{Node A: } R_{AX} = F_{AB}(3/5) - V_{AB}(4/5) = -qL \text{ Leftward ;}$$

$$R_{AY} = F_{AB}(4/5) + V_{AB}(3/5) = 0.08qL \text{ Upward}$$

By exploiting anti-symmetry conditions, end moments, end shear forces, end axial forces and support reactions for the right half of the structure can readily be obtained as follows:

$$M_{DC} = M_{AB} = 0 ; M_{CD} = M_{BA} = 2.239qL^2 ; M_{CB} = M_{BC} = -0.738qL^2$$

$$V_{DC} = -V_{AB} = -0.848qL ; V_{CD} = -V_{BA} = 0.048qL ; V_{CB} = -V_{BC} = 0.920qL$$

$$F_{DC} = F_{AB} = -0.536qL ; F_{CD} = F_{BA} = -0.064qL ; F_{CB} = F_{BC} = 0$$

$$R_{DX} = R_{AX} = -qL ; R_{DY} = -R_{AY} = -0.08qL$$

Now, responses of the original structure can be obtained by superposing those for symmetric and anti-symmetric structures. Final results are given below:

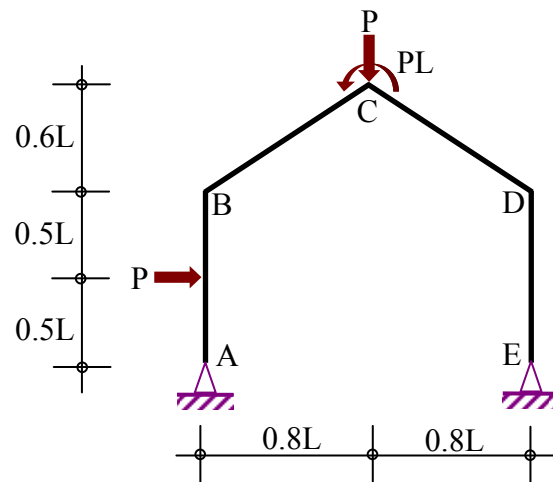
$$M_{AB} = 0 ; M_{BA} = 5qL^2 / 32 + 2.239qL^2 = 2.395qL^2 ; M_{BC} = 43qL^2 / 32 - 0.738qL^2 = 0.606qL^2 ;$$

$$M_{CB} = -43qL^2 / 32 - 0.738qL^2 = -2.082qL^2 ; M_{CD} = -5qL^2 / 32 + 2.239qL^2 = 2.083qL^2 ; M_{DC} = 0 ;$$

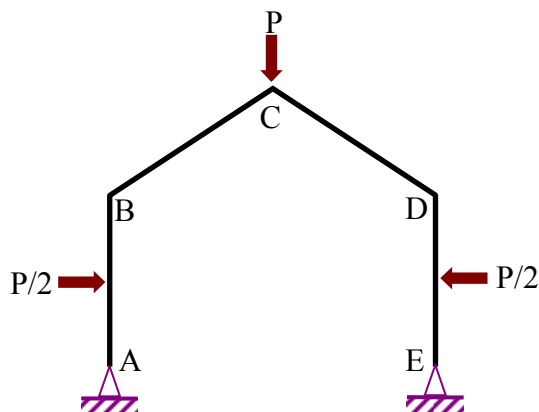
$$V_{AB} = 69qL / 160 + 0.848qL = 1.279qL ; V_{BA} = 59qL / 160 - 0.048qL = 0.321qL ;$$

$$\begin{aligned}
 V_{BC} &= 5qL / 4 - 0.920qL = 0.330qL ; & V_{CB} &= 5qL / 4 + 0.920qL = 2.170qL ; \\
 V_{CD} &= 59qL / 160 + 0.048qL = 0.417qL ; & V_{DC} &= 69qL / 160 - 0.848qL = -0.417qL ; \\
 F_{AB} &= 1593qL / 640 - 0.536qL = 1.953qL ; & F_{BA} &= -1977qL / 640 - 0.064qL = -3.153qL ; \\
 F_{BC} &= 275qL / 128 + 0 = 2.148qL ; & F_{CB} &= -275qL / 128 + 0 = -2.148qL ; \\
 F_{CD} &= 1977qL / 640 - 0.064qL = 3.025qL ; & F_{AC} &= -1593qL / 640 - 0.536qL = -3.025qL ; \\
 R_{AX} &= 147qL / 128 - qL = 0.148qL \text{ Rightward} ; & R_{AY} &= 9qL / 4 + 0.08qL = 2.33qL \text{ Upward} \\
 R_{DX} &= -147qL / 128 - qL = -2.148qL \text{ Leftward} ; & R_{DY} &= 9qL / 4 - 0.08qL = 2.17qL \text{ Upward}
 \end{aligned}$$

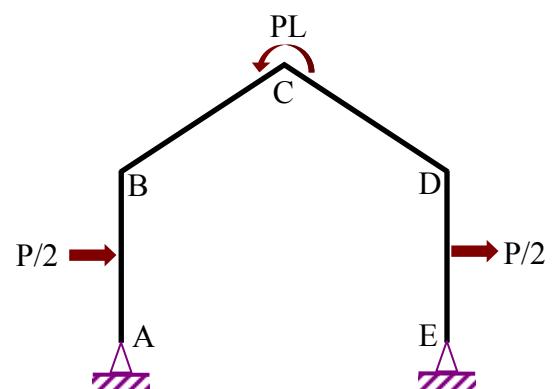
**Example 12.16** Use the slope-deflection method along with symmetric and anti-symmetric features to analyze a frame subjected to external loads shown below. The flexural rigidity  $EI$  is assumed to be constant throughout.



**Solution** Since the given structure is geometrically symmetric subjected to a set of general loadings, it can be decomposed into symmetric and anti-symmetric structures shown below. Analysis of each structure by the method of slope-deflection is demonstrated below.



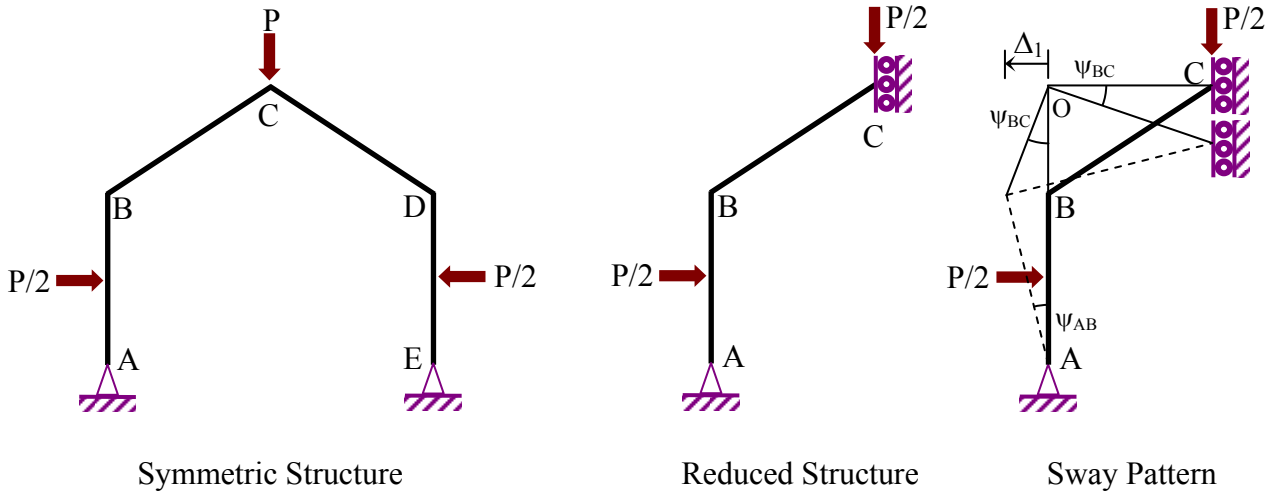
Symmetric Structure



Anti-symmetric Structure

**Symmetric Structure:** Since the structure is subjected to a set of symmetric loadings and the plane of symmetry passes through the node, an equivalent half of the structure illustrated below can be used in the analysis. By discretizing such reduced structure into two members AB and BC with three nodes A, B, and C, the number of rotational degrees of freedom becomes  $N_r = 3 + 0 - 1 = 2$  (i.e.  $\theta_A$

and  $\theta_B$ ) whereas the number of sway degrees of freedom is equal to  $N_s = 2(3) - 3 - 2 = 1$ . Therefore, the number of primary unknowns is equal to  $2 + 1 = 3$ . By noting that the bending moment at node A is prescribed, the modified slope-deflection (12.74) or (12.76) can be applied to the member AB and the number of primary unknowns can be reduced from 3 to 2 by eliminating the rotation  $\theta_A$ . The sway pattern associated with the sway degree of freedom (the horizontal displacement at node B, denoted by  $\Delta_1$ ) can be sketched as shown below.



The sway angle of members AB and BC can be expressed in terms of the sway degree of freedom  $\Delta_1$  by using the ICR concept as follows. The ICR of the member AB is located at node A (since it is a fixed point) whereas the ICR of the member BC is obtained from the intersection of the line AB and the horizontal line emanating from the node C denoted by a point O (since nodes B and C are allowed to move only in horizontal and vertical directions, respectively). By using the relation (12.68) along with the known ICR and known horizontal displacement at node B in terms of  $\Delta_1$ , the sway angle of member AB and BC are obtained as  $\Delta_1/L$  and  $-5\Delta_1/3L$ , respectively. The modified slope-deflection equation for the member AB (a member with prescribed bending moment at one end) and standard slope-deflection equations for the member BC are given below.

**Member AB** For this member,  $M_{A0} = 0$ ,  $\psi_{AB} = \Delta_1/L$ ,  $FEM_{AB} = (P/2)(L)/8 = PL/16$ ,  $FEM_{BA} = -(P/2)(L)/8 = -PL/16$  and the modified fixed-end moment at the end B is given by

$$\overline{FEM}_{BA} = FEM_{BA} - \frac{1}{2}FEM_{AB} + \frac{1}{2}M_{A0} = -\frac{PL}{16} - \frac{1}{2}\left(\frac{PL}{16}\right) + \frac{1}{2}(0) = -\frac{3PL}{32}$$

The modified slope-deflection equations become

$$M_{BA} = \frac{3EI}{L}\theta_B - \frac{3EI}{L}\left(\frac{\Delta_1}{L}\right) - \frac{3PL}{32} = \frac{3EI\theta_B}{L} - \frac{3EI\Delta_1}{L^2} - \frac{3PL}{32} \quad (e12.16.1)$$

The rotation at the end A is given by

$$\theta_A = -\frac{1}{2}\theta_B + \frac{3}{2}\left(\frac{\Delta_1}{L}\right) - \frac{L}{4EI}\left(\frac{PL}{16} - 0\right) = -\frac{1}{2}\theta_B + \frac{3\Delta_1}{2L} - \frac{PL^2}{64EI} \quad (e12.16.2)$$

**Member BC** For this member,  $\theta_C = 0$ ,  $\psi_{BC} = -5\Delta_1/3L$ ,  $FEM_{BC} = FEM_{CB} = 0$ , and the slope-deflection equations become

$$M_{BC} = \frac{2EI}{L}(2\theta_B + 0) - \frac{6EI}{L}\left(-\frac{5\Delta_1}{3L}\right) + 0 = \frac{4EI\theta_B}{L} + \frac{10EI\Delta_1}{L^2} \quad (e12.16.3)$$

$$M_{CB} = \frac{2EI}{L}(\theta_B + 0) - \frac{6EI}{L}\left(-\frac{5\Delta_1}{3L}\right) + 0 = \frac{2EI\theta_B}{L} + \frac{10EI\Delta_1}{L^2} \quad (\text{e12.16.4})$$

The moment equilibrium equation at node B is obtained as follows:

$$\begin{aligned} [\Sigma M_{\text{node B}} = 0] &\Rightarrow M_{BA} + M_{BC} = 0 \\ &\Rightarrow \left(\frac{3EI\theta_B}{L} - \frac{3EI\Delta_1}{L^2} - \frac{3PL}{32}\right) + \left(\frac{4EI\theta_B}{L} + \frac{10EI\Delta_1}{L^2}\right) = 0 \\ &\Rightarrow \frac{7EI\theta_B}{L} + \frac{7EI\Delta_1}{L^2} = \frac{3PL}{32} \end{aligned} \quad (\text{e12.16.5})$$

The equilibrium equation associated with the sway degree of freedom is obtained by using the principle of virtual work along with the sway pattern being chosen as the virtual displacement:

$$\begin{aligned} \delta W_{\text{ext}} &= -(P/2)|\psi_{AB}|(L/2) + (P/2)|\psi_{BC}|(0.8L) = 5P\Delta_1/12 \\ \delta W_{\text{int}} &= -(M_{AB} + M_{BA})\psi_{AB} - (M_{BC} + M_{CB})\psi_{BC} = -M_{BA}\Delta_1/L + 5(M_{BC} + M_{CB})\Delta_1/3L \\ \delta W_{\text{ext}} = \delta W_{\text{int}} &\Rightarrow 5P\Delta_1/12 = -M_{BA}\Delta_1/L + 5(M_{BC} + M_{CB})\Delta_1/3L \\ &\Rightarrow -M_{BA} + 5(M_{BC} + M_{CB})/3 = 5PL/12 \\ &\Rightarrow \frac{7EI\theta_B}{L} + \frac{109EI\Delta_1}{3L^2} = \frac{31PL}{96} \end{aligned} \quad (\text{e12.16.6})$$

By solving a system of linear equations (e12.16.5) and (e12.16.6), we obtain

$$\frac{EI\theta_B}{L} = \frac{5PL}{896} \quad \text{and} \quad \frac{EI\Delta_1}{L^2} = \frac{PL}{128}$$

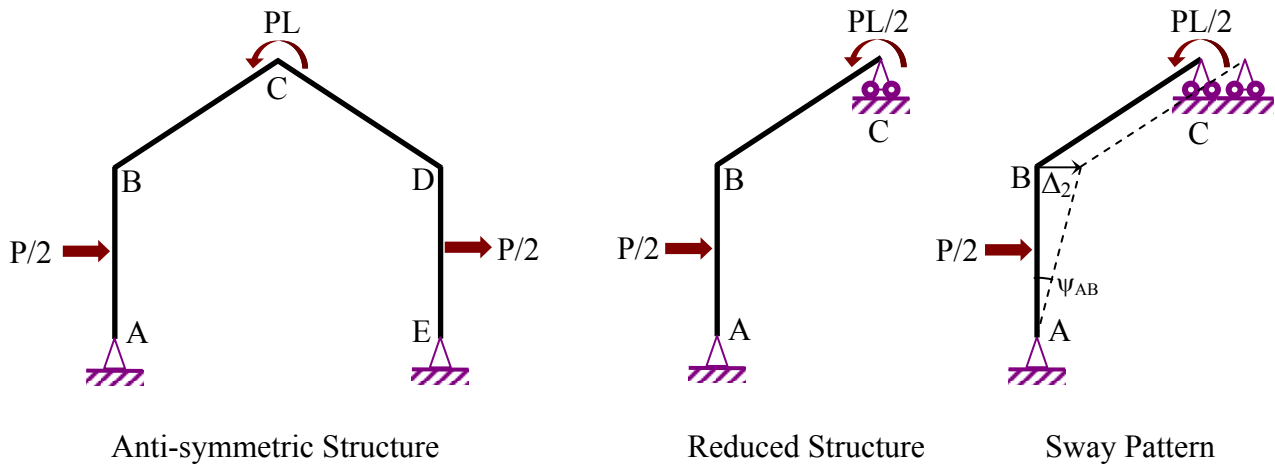
The rotation at the node A is obtained from (e12.16.2) as

$$\theta_A = -\frac{1}{2}\theta_B + \frac{3\Delta_1}{2L} - \frac{PL^2}{64EI} = -\frac{1}{2}\left(\frac{5PL^2}{896EI}\right) + \frac{3}{2L}\left(\frac{PL^3}{128EI}\right) - \frac{PL^2}{64EI} = -\frac{3PL^2}{448EI}$$

Next, the end moments, end shear forces, and end axial forces of members AB and BC can be obtained from the slope-deflection equations (e12.16.1) and (e12.16.3)-(e12.16.4) and equilibrium of both members and node B whereas reactions at the pinned support can be computed from equilibrium at node A (see similar procedure in previous examples). Results obtained for the reduced structure are identical to those of the left half of the symmetric structure and results for its right half can readily be deduced from the symmetry conditions (e.g.  $\theta_D = -\theta_A$ ,  $\theta_C = -\theta_B$ ,  $M_{BA} = -M_{CD}$ ,  $M_{BC} = -M_{CB}$ ,  $R_{AX} = -R_{DX}$ ,  $R_{AY} = R_{DY}$ , etc.).

**Anti-symmetric Structure:** Since this structure is subjected to a set of anti-symmetric loadings and the plane of symmetry passes through the node, an equivalent half of the structure illustrated below can be used in the analysis. By discretizing such reduced structure into two members AB and BC with three nodes A, B, and C, the number of rotational degrees of freedom becomes  $N_r = 3 + 0 - 0 = 3$  (i.e.  $\theta_A$ ,  $\theta_B$  and  $\theta_C$ ) whereas the number of sway degrees of freedom is equal to  $N_s = 2(3) - 3 - 2 = 1$ . Therefore, the number of primary unknowns is equal to  $3 + 1 = 4$ . By further using the modified slope-deflection equation for member AB and BC due to the prescribed bending moment at node A and node C, the number of primary unknowns can be reduced from 4 to 2; the rotation  $\theta_A$  and  $\theta_C$

can be eliminated from a set of primary unknowns. The sway pattern associated with the sway degree of freedom (the horizontal displacement at node B, denoted by  $\Delta_2$ ) can be sketched as shown below.



The sway angle of members AB and BC can be expressed in terms of the sway degree of freedom  $\Delta_2$  by using the ICR concept as follows. The ICR of the member AB is located at node A (since it is a fixed point) whereas the ICR of the member BC is located at infinity (since both nodes B and C are allowed to move only in horizontal directions). By using the relation (12.68) along with the known ICR and known horizontal displacement at node B in terms of  $\Delta_2$ , the sway angle of member AB and BC are obtained as  $-\Delta_2/L$  and 0, respectively. The modified slope-deflection equations for the member AB (a member with prescribed bending moment at node A) and the member BC (a member with prescribed bending moment at node C) are given below.

**Member AB** For this member,  $M_{A0} = 0$ ,  $\psi_{AB} = -\Delta_2/L$ ,  $FEM_{AB} = (P/2)(L)/8 = PL/16$ ,  $FEM_{BA} = -(P/2)(L)/8 = -PL/16$  and the modified fixed-end moment at the end B is given by

$$\overline{FEM}_{BA} = FEM_{BA} - \frac{1}{2} FEM_{AB} + \frac{1}{2} M_{A0} = -\frac{PL}{16} - \frac{1}{2} \left( \frac{PL}{16} \right) + \frac{1}{2} (0) = -\frac{3PL}{32}$$

The modified slope-deflection equations become

$$M_{BA} = \frac{3EI}{L} \theta_B - \frac{3EI}{L} \left( -\frac{\Delta_2}{L} \right) - \frac{3PL}{32} = \frac{3EI\theta_B}{L} + \frac{3EI\Delta_2}{L^2} - \frac{3PL}{32} \quad (e12.16.7)$$

The rotation at the end A is given by

$$\theta_A = -\frac{1}{2} \theta_B + \frac{3}{2} \left( -\frac{\Delta_2}{L} \right) - \frac{L}{4EI} \left( \frac{PL}{16} - 0 \right) = -\frac{1}{2} \theta_B - \frac{3\Delta_2}{2L} - \frac{PL^2}{64EI} \quad (e12.16.8)$$

**Member BC** For this member,  $M_{C0} = PL/2$ ,  $\psi_{BC} = 0$ ,  $FEM_{BC} = FEM_{CB} = 0$  and the modified fixed-end moment at the end B is given by

$$\overline{FEM}_{BC} = FEM_{BC} - \frac{1}{2} FEM_{CB} + \frac{1}{2} M_{C0} = 0 - \frac{1}{2} (0) + \frac{1}{2} \left( \frac{PL}{2} \right) = \frac{PL}{4}$$

The modified slope-deflection equations become

$$M_{BA} = \frac{3EI}{L} \theta_B - \frac{3EI}{L} (0) + \frac{PL}{4} = \frac{3EI\theta_B}{L} + \frac{PL}{4} \quad (e12.16.9)$$

The rotation at the end C is given by

$$\theta_C = -\frac{1}{2}\theta_B + \frac{3}{2}(0) - \frac{L}{4EI}\left(0 - \frac{PL}{2}\right) = -\frac{1}{2}\theta_B + \frac{PL^2}{8EI} \quad (\text{e12.16.10})$$

The moment equilibrium equation at node B is obtained as follows:

$$\begin{aligned} [\Sigma M_{\text{node B}} = 0] &\Rightarrow M_{BA} + M_{BC} = 0 \\ &\Rightarrow \left(\frac{3EI\theta_B}{L} + \frac{3EI\Delta_2}{L^2} - \frac{3PL}{32}\right) + \left(\frac{3EI\theta_B}{L} + \frac{PL}{4}\right) = 0 \\ &\Rightarrow \frac{6EI\theta_B}{L} + \frac{3EI\Delta_2}{L^2} = -\frac{5PL}{32} \end{aligned} \quad (\text{e12.16.11})$$

The equilibrium equation associated with the sway degree of freedom is obtained by using the principle of virtual work along with the sway pattern being chosen as the virtual displacement:

$$\begin{aligned} \delta W_{\text{ext}} &= (P/2)|\psi_{AB}|(L/2) = P\Delta_2/4 \\ \delta W_{\text{int}} &= -(M_{AB} + M_{BA})\psi_{AB} - (M_{BC} + M_{CB})\psi_{BC} = M_{BA}\Delta_2/L \\ \delta W_{\text{ext}} = \delta W_{\text{int}} &\Rightarrow P\Delta_2/4 = M_{BA}\Delta_2/L \Rightarrow M_{BA} = PL/4 \\ &\Rightarrow \frac{3EI\theta_B}{L} + \frac{3EI\Delta_2}{L^2} = \frac{11PL}{32} \end{aligned} \quad (\text{e12.16.12})$$

By solving a system of linear equations (e12.16.11) and (e12.16.12), we obtain

$$\frac{EI\theta_B}{L} = -\frac{PL}{6} \quad \text{and} \quad \frac{EI\Delta_2}{L^2} = \frac{9PL}{32}$$

The rotations at nodes A and C are obtained from (e12.16.8) and (e12.16.10) as

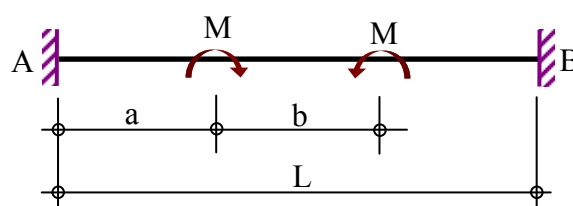
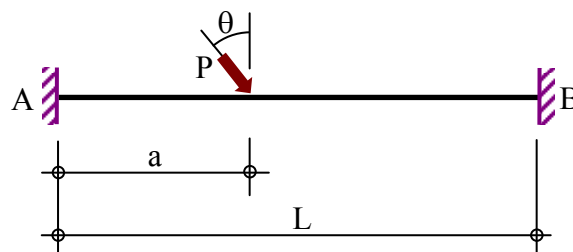
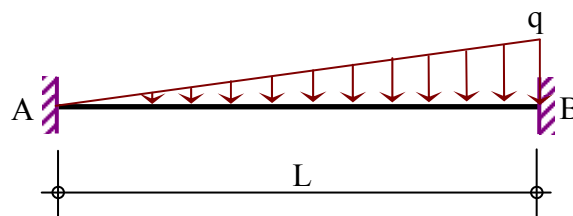
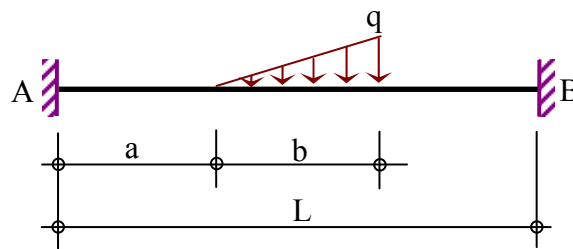
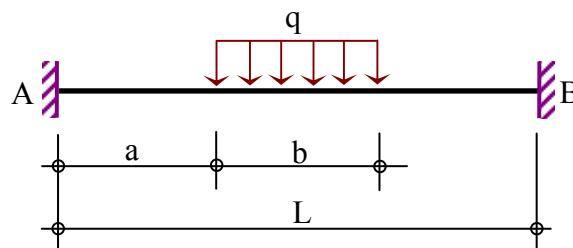
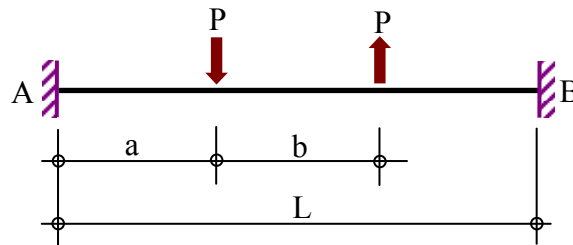
$$\begin{aligned} \theta_A &= -\frac{1}{2}\theta_B - \frac{3\Delta_2}{2L} - \frac{PL^2}{64EI} = -\frac{1}{2}\left(-\frac{PL^2}{6EI}\right) - \frac{3}{2L}\left(\frac{9PL^3}{32EI}\right) - \frac{PL^2}{64EI} = -\frac{17PL^2}{48EI} \\ \theta_A &= -\frac{1}{2}\theta_B + \frac{PL^2}{8EI} = -\frac{1}{2}\left(-\frac{PL^2}{6EI}\right) + \frac{PL^2}{8EI} = \frac{5PL^2}{24EI} \end{aligned}$$

Next, the end moments, end shear forces, and end axial forces of members AB and BC can be obtained from the slope-deflection equations (e12.16.7) and (e12.16.9) and equilibrium of both members and node B whereas reactions at the pinned support can be computed from equilibrium at node A. Results obtained for the reduced structure are identical to those of the left half of the anti-symmetric structure and results for its right half can readily be deduced from the anti-symmetry conditions (e.g.  $\theta_D = \theta_A$ ,  $\theta_C = \theta_B$ ,  $M_{BA} = M_{CD}$ ,  $M_{BC} = M_{CB}$ ,  $R_{AX} = R_{DX}$ ,  $R_{AY} = -R_{DY}$ , etc.).

Finally, responses of the original structure (e.g. rotations, end moments, end shear forces, end axial forces, support reactions, AFD, SFD, BMD, etc.) can be obtained by superposing those for symmetric and anti-symmetric structures.

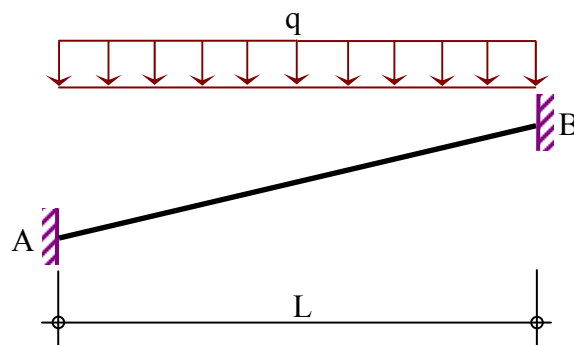
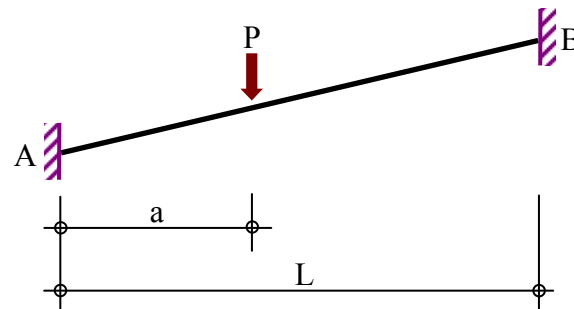
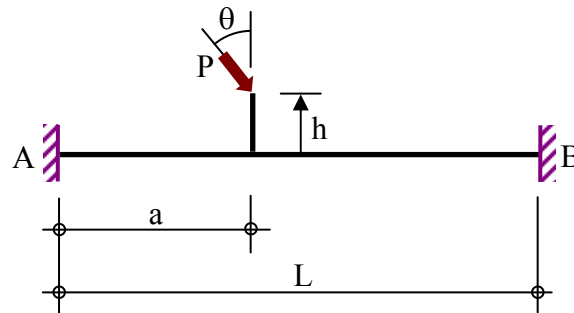
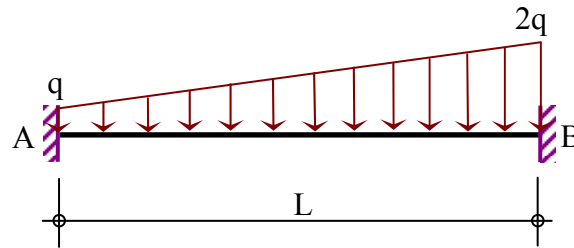
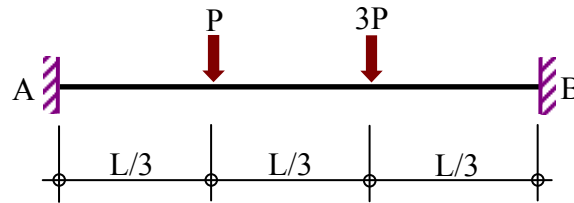
## EXERCISES

**Problem 1:** Use either equations (12.17)-(12.8) or equations (12.24)-(12.25) to compute the fixed-end moment of a member subjected to applied loads shown below.

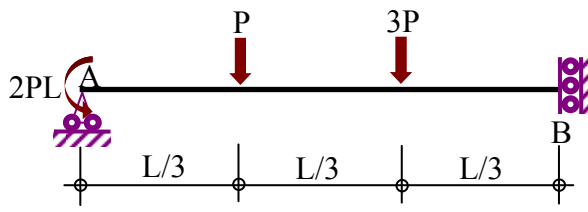
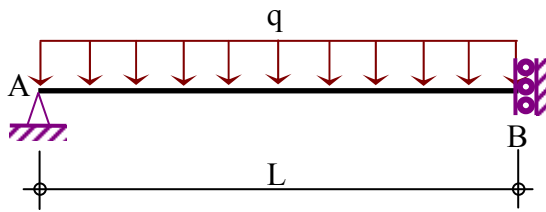
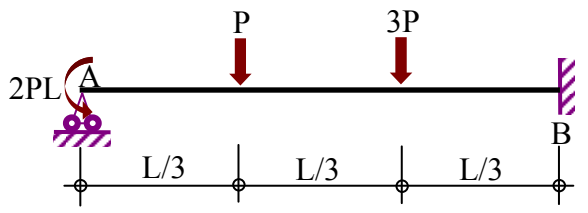
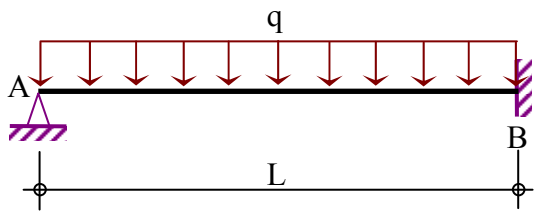
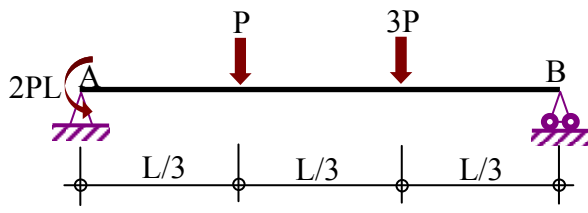
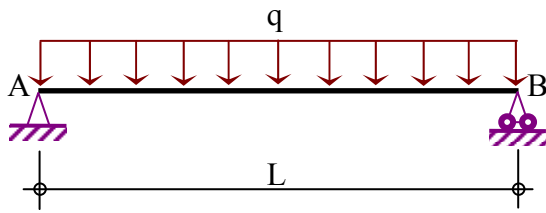




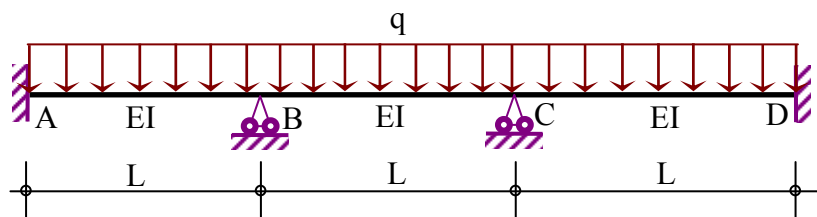
**Problem 2:** Use results from Table 12.1 along with the method of superposition (if needed) to determine the fixed-end moments for members shown below.



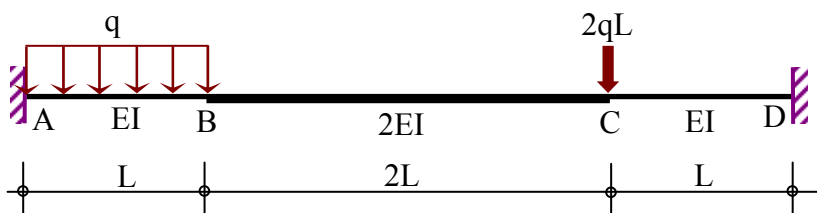
**Problem 3:** Use slope-deflection equations to determine the end rotations and/or end deflections of single-span beams subjected to external loads shown below. The flexural rigidity of the member  $EI$  is assumed to be constant throughout.



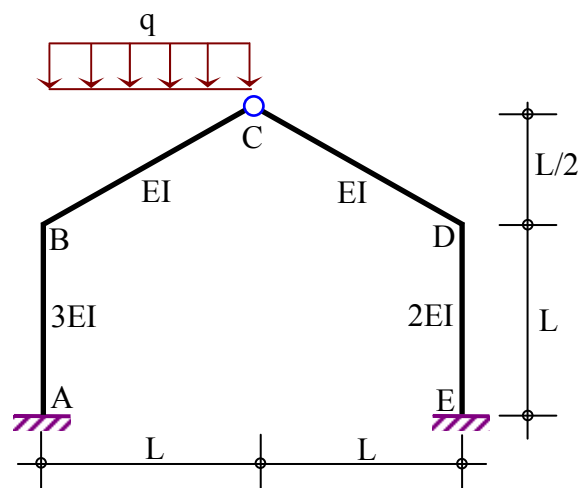
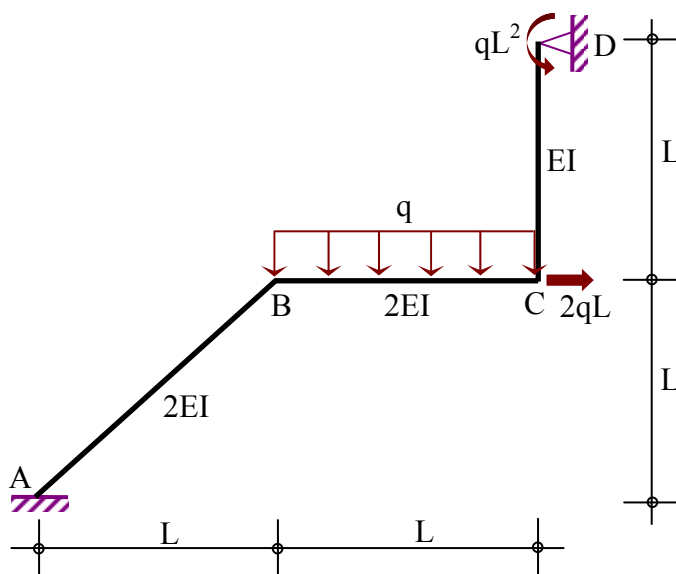
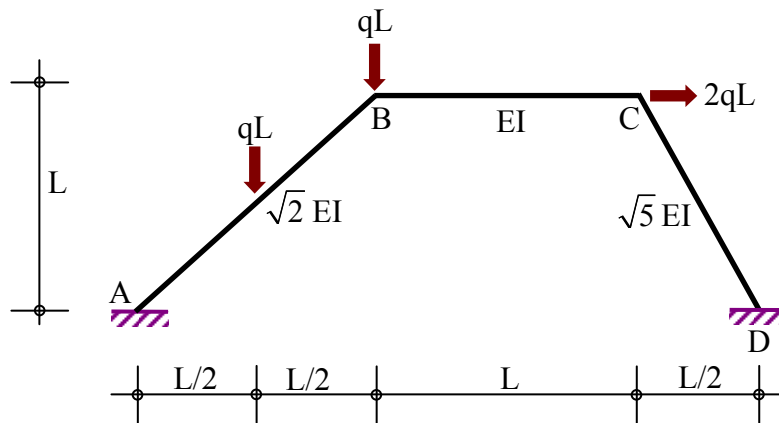
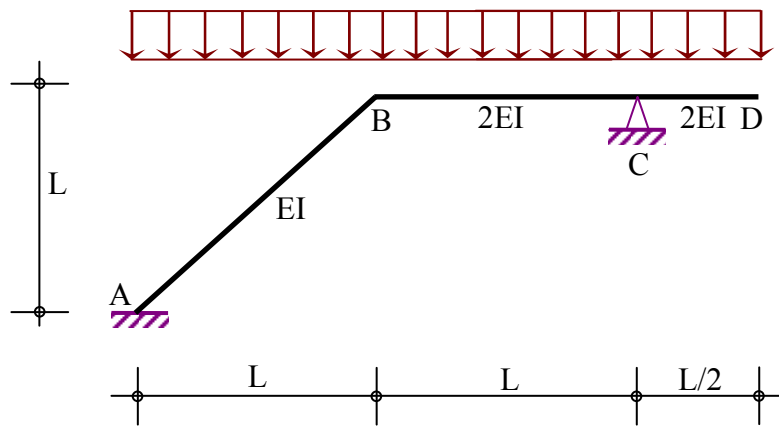
**Problem 4:** Use the slope-deflection method to analyze the prismatic beam of flexural rigidity  $EI$  and subjected to a uniformly distributed load  $q$  as shown below. The roller supports at B and C are subjected to downward settlement equal to  $\Delta_0$ . Also compute all support reactions and draw SFD and BMD.

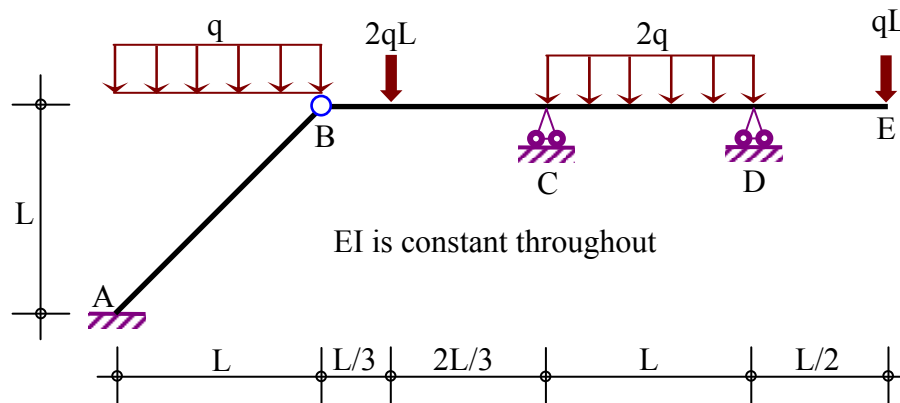


**Problem 5:** Use the slope-deflection method to analyze the beam subjected to external loads as shown below. Also compute all support reactions and draw SFD and BMD.



**Problem 6:** Use the slope-deflection method to analyze frames subjected to external loads as shown below. Also compute all support reactions and draw AFD, SFD and BMD.





**Problem 7:** Use the slope-deflection method along with the symmetry and anti-symmetry features of the structure to analyze geometrically symmetric frames subjected to external loads shown below.

